IV.2 Operator correlators on the contour

Goal (as always): Compute $O(t)$

Expansion of time-ordered exponentials in Taylor series yields strings of operators $\equiv$ operator correlators

$$\hat{k}(z_1, \dots, z_n) = \mathcal{T}\left\{\hat{O}(z_1) \dots \hat{O}(z_n)\right\}.$$

Alternatively we could use the EOM for $\hat{O}_H(z)$, trace with $\hat{\rho}$ and solve the differential equations. But this leads to new operators, which requires more differential equations, etc.

So we are back to calculating the trace of strings of operators in the contour Heisenberg picture, for instance $\hat{n}_H(x',z)\,\hat{\psi}_H(x,z) = \mathcal{T}\left\{n_H(x',z^+)\,\hat{\psi}_H(x,z)\right\}$ where $z^+ > z$ infinitesimally.

end Lecture 7

$\Rightarrow$ need to find relations for operator correlators.

Abbreviation: $\hat{O}_j \equiv \hat{O}_j(z_j)$

Example: $\mathcal{T}\left\{\hat{O}_1\,\hat{O}_2\right\} = \underbrace{\Theta(z_1,z_2)}_{\text{Heaviside fct. on } \gamma}\hat{O}_1\,\hat{O}_2 + \Theta(z_2,z_1)\,\hat{O}_2\,\hat{O}_1$

$$\Rightarrow \frac{d}{dz_1}\mathcal{T}\left\{\hat{O}_1\,\hat{O}_2\right\} = \delta(z_1,z_2)\left[\hat{O}_1,\hat{O}_2\right]_- + \mathcal{T}\left\{\left(\frac{d}{dz_1}\hat{O}_1\right)\hat{O}_2\right\}$$

with Dirac delta function on γ

$$\delta(z_1,z_2) \equiv \frac{d}{dz_1}\Theta(z_1,z_2) \equiv -\frac{d}{dz_2}\Theta(z_1,z_2).$$

The δ-fct. is zero everywhere except for $z_1 = z_2$,

and $\displaystyle\int_{z_i}^{z_f} d\bar{z}\ \delta(z,\bar{z})\, \hat{A}(\bar{z}) = \hat{A}(z)$.

Most important case: $\hat{O}_1$ and $\hat{O}_2$ field operators.

For bosons the structure with $[\hat{O}_1, \hat{O}_2]_-$ at equal times is nice. For fermions we prefer an anti-commutator to obtain simple expressions. We define time-ordering for fermionic operators

$$\mathcal{T}\{\hat{O}_1 \hat{O}_2\} \equiv \Theta(z_1, z_2)\, \hat{O}_1 \hat{O}_2 - \Theta(z_2, z_1)\, \hat{O}_2 \hat{O}_1$$

which gives

$$\frac{d}{dz_1} \mathcal{T}\{\hat{O}_1 \hat{O}_2\} = \delta(z_1, z_2)\,[\hat{O}_1, \hat{O}_2]_+ + \mathcal{T}\left\{\left(\frac{d}{dz_1}\hat{O}_1\right)\hat{O}_2\right\}.$$

Generalized definition:

$$\mathcal{T}\{\hat{O}_1 \ldots \hat{O}_n\} = \sum_P (\pm)^P\, \Theta_n(z_{P(1)}, \ldots z_{P(n)})\, \hat{O}_{P(1)} \ldots \hat{O}_{P(n)}$$

$+$: bosons

$-$: fermions

n-time theta function: $\Theta_n(z_1, \ldots, z_n) \equiv \Theta(z_1, z_2) \ldots \Theta(z_{n-1}, z_n)$.

$$\Rightarrow \mathcal{T}\{\hat{O}_1 \ldots \hat{O}_n\} = (\pm)^P\, \mathcal{T}\{\hat{O}_{P(1)} \ldots \hat{O}_{P(n)}\}$$

Bosonic operators commute with the contour-ordered product

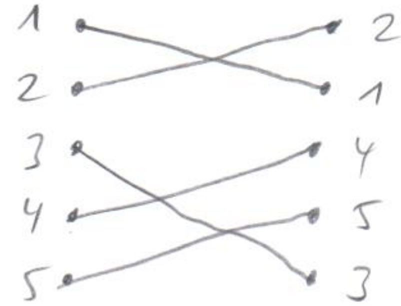
Fermionic " anti-commute ——— " ———

For many fermionic operators there is a nice graphical way to find the sign of a permutation.

E.g., contour-ordering with

$$z_2 > z_1 > z_4 > z_5 > z_3 .$$

Draw



Count the number of crossings.

Even $\rightarrow$ $+$

odd $\rightarrow$ $-$

Here: $\#\text{crossings} = 3 \equiv n_c$

$\Rightarrow \text{sign}(P) = (\pm)^P = (\pm 1)^{n_c}$

$$\Rightarrow \quad \mathcal{T}\{\hat O_1 \hat O_2 \hat O_3 \hat O_4 \hat O_5\} = -\hat O_2 \hat O_1 \hat O_4 \hat O_5 \hat O_3$$

with $P(1,2,3,4,5) = (2,1,4,5,3)$.

$$(*) \quad \frac{d}{dz_k}\mathcal{T}\{\hat O_1 \dots \hat O_n\} = \partial^{\theta}_{z_k}\mathcal{T}\{\hat O_1 \dots \hat O_n\} + \mathcal{T}\left\{\hat O_1 \dots \hat O_{k-1}\left(\frac{d}{dz_k}\hat O_k\right)\hat O_{k+1}\dots \hat O_n\right\}$$

$$\partial^{\theta}_{z_k}\mathcal{T}\{\hat O_1 \dots \hat O_n\} \equiv \sum_P (\pm)^P \left(\frac{d}{dz_k}\theta_n(z_{P(1)},\dots,z_{P(n)})\right)\hat O_{P(1)}\dots \hat O_{P(n)}$$

More explicitely one can show that $\cdots \hat O_{l-1}\hat O_{l+1}$

$$\partial^{\theta}_{z_k}\mathcal{T}\{\hat O_1 \dots \hat O_n\} = \sum_{l=1}^{k-1}(\pm)^{k-l}\delta(z_k,z_l)\,\mathcal{T}\left\{\hat O_1 \dots \hat O_{l-1}\left[\hat O_k,\hat O_l\right]_{\mp}\hat O_{k+1}\dots \hat O_n\right\}$$

$$(**) \qquad + \sum_{l=k+1}^{n}(\pm)^{l-k-1}\delta(z_k,z_l)\,\mathcal{T}\left\{\hat O_1 \dots \hat O_{k-1}\left[\hat O_k,\hat O_l\right]_{\mp}\hat O_{k+1}\dots \hat O_{l-1}\hat O_{l+1}\dots \hat O_n\right\}$$

(**) and (*) are the n-operator generalization of the EoM for 2 operators.

For the example of 5 operators:

$$\frac{d}{dz_3}\mathcal{T}\left\{\hat{O}_1\hat{O}_2\hat{O}_3\hat{O}_4\hat{O}_5\right\} = \delta(z_3,z_1)\,\mathcal{T}\left\{\hat{O}_2\left[\hat{O}_3,\hat{O}_1\right]_\mp\hat{O}_4\hat{O}_5\right\}$$
$$\pm\,\delta(z_3,z_2)\,\mathcal{T}\left\{\hat{O}_1\left[\hat{O}_3,\hat{O}_2\right]_\mp\hat{O}_4\hat{O}_5\right\}$$
$$+\,\delta(z_3,z_4)\,\mathcal{T}\left\{\hat{O}_1\hat{O}_2\left[\hat{O}_3,\hat{O}_4\right]_\mp\hat{O}_5\right\}$$
$$\pm\,\delta(z_3,z_5)\,\mathcal{T}\left\{\hat{O}_1\hat{O}_2\left[\hat{O}_3,\hat{O}_5\right]_\pm\hat{O}_4\right\}$$
$$+\,\mathcal{T}\left\{\hat{O}_1\hat{O}_2\left(\frac{d}{dz_3}\hat{O}_3\right)\hat{O}_4\hat{O}_5\right\}$$

with signs determined by the required number of interchanges to shift $\hat{O}_\ell$ $(\ell=1,2,4,5)$ directly after $\hat{O}_3$.

Specifically for field operators in the contour Heisenberg picture,

$$\left[\hat{O}_k(z),\,\hat{O}_\ell(z)\right]_\mp = \underbrace{c_{k\ell}(z)}_{\text{number}}\,\hat{\mathbb{1}}\,.$$

The $\hat{\mathbb{1}}$ commutes with all Fock space operators and can be moved outside the contour-ordered product:

$$\partial^\theta_{z_k}\mathcal{T}\left\{\hat{O}_1\ldots\hat{O}_n\right\} = \sum_{\ell=1}^{k-1}(\pm)^{k-\ell}\,\delta(z_k,z_\ell)\left[\hat{O}_k,\hat{O}_\ell\right]_\mp\mathcal{T}\left\{\hat{O}_1\ldots\overset{\sqcap}{\hat{O}_\ell}\ldots\overset{\sqcap}{\hat{O}_k}\ldots\hat{O}_n\right\}$$
$$+\sum_{\ell=k+1}^{n}(\pm)^{\ell-k-1}\,\delta(z_k,z_\ell)\left[\hat{O}_k,\hat{O}_\ell\right]_\mp\mathcal{T}\left\{\hat{O}_1\ldots\overset{\sqcap}{\hat{O}_k}\ldots\overset{\sqcap}{\hat{O}_\ell}\ldots\hat{O}_n\right\}$$

"$\sqcap$": operator missing from list

We work out the time derivative for a case of 4 field operators:

Define
$$i \equiv x_i, z_i$$
$$j \equiv x_j, z_j \qquad \text{etc.}$$
$$i' \equiv x_i', z_i'$$
$$\bar{i} \equiv \bar{x}_i, \bar{z}_i$$

$$\delta(j;k) \equiv \delta(z_j, z_k)\, \delta(x_j - x_k)$$

Then we have
$$\frac{d}{dz_2} \mathcal{T}\left\{\hat{\varphi}_H(1)\, \hat{\varphi}_H(2)\, \hat{\varphi}^+(3)\, \hat{\varphi}_H^+(4)\right\} = \mathcal{T}\left\{\hat{\varphi}_H(1)\left(\frac{d}{dz_2}\hat{\varphi}_H(2)\right) \hat{\varphi}_H^+(3)\, \hat{\varphi}_H^+(4)\right\}$$
$$+ \delta(2;3)\, \mathcal{T}\left\{\hat{\varphi}_H(1)\, \hat{\varphi}_H^+(4)\right\}$$
$$\pm \delta(2;4)\, \mathcal{T}\left\{\hat{\varphi}_H(1)\, \hat{\varphi}^+(3)\right\}.$$

Let us now define the n-particle correlator (Green's function) as
$$\hat{G}_n(1,\dots,n; 1',\dots,n') \equiv \frac{1}{i^n} \mathcal{T}\left\{\hat{\varphi}_H(1)\dots\hat{\varphi}_H(n)\, \hat{\varphi}_H^+(n')\dots \hat{\varphi}_H^+(1')\right\}$$
$$\hat{G}_0 \equiv \mathbb{1} \quad (n=0)$$

We identify $\hat{O}_j = \begin{cases} \hat{\varphi}_H(j) & , j = 1,\dots,n \\ \hat{\varphi}_H^+((2n-j+1)') & , j = n+1,\dots,2n \end{cases}$

and find

$$i \frac{d}{dz_k} \hat{G}_n(1, \dots, n; 1', \dots, n')$$

$$= \frac{1}{i^n} \mathcal{T}\left\{\hat{\psi}_H(1) \dots \left(i \frac{d}{dz_k} \hat{\psi}_H(k)\right) \dots \hat{\psi}_H(n) \hat{\psi}_H^+(n') \dots \hat{\psi}_H^+(1')\right\}$$

$$+ \sum_{j=1}^{n} (\pm)^{k+j} \delta(k, j') \hat{G}_{n-1}(1, \dots \sqcap_k \dots, n; 1', \dots, \sqcap_{j'} \dots, n')$$

and

$$-i \frac{d}{dz_k'} \hat{G}_n(1, \dots, n; 1', \dots, n')$$

$$= \frac{1}{i^n} \mathcal{T}\left\{\hat{\psi}_H(1) \dots \hat{\psi}_H(n) \hat{\psi}_H^+(n') \dots \left(-i \frac{d}{dz_k'} \hat{\psi}_H^+(k')\right) \dots \hat{\psi}_H^+(1')\right\}$$

$$+ \sum_{j=1}^{n} (\pm)^{k+j} \delta(j; k') \hat{G}_{n-1}(1, \dots \sqcap_j \dots, n; 1', \dots, \sqcap_{k'}, \dots, n')$$

$[(n-j) + (n-k)$ interchanges, and $(\pm)^{(n-j)+(n-k)} = (\pm)^{k+j}]$.

Now we assume that $\hat{h}$ is diagonal in spin space

$$\Rightarrow \langle x_1 | \hat{h}(z_1) | x_2 \rangle = h(1)\, \delta(x_1 - x_2) = \delta(x_1 - x_2)\, h(2)$$

$\Rightarrow$ EOMs

$$i \frac{d}{dz_k} \hat{\psi}_H(k) = h(k) \hat{\psi}_H(k) + \int d\bar{1}\, v(k; \bar{1})\, \hat{n}_H(\bar{1})\, \hat{\psi}_H(k)$$

$$-i \frac{d}{dz_k'} \hat{\psi}_H^+(k') = \hat{\psi}_H^+(k') h(k') + \int d\bar{1}\, v(k'; \bar{1})\, \hat{\psi}_H^+(k')\, \hat{n}_H(\bar{1})$$

with $v(i; j) \equiv \delta(z_i, z_j)\, v(x_i, x_j, z_i)$.

Inside the $\mathcal{T}$ sign we can write

$$\mathcal{T}\{\ldots \hat{n}_H(\bar{1})\, \hat{\psi}_H(k) \ldots\} = \pm\, \mathcal{T}\{\ldots \hat{\psi}_H(k)\, \hat{\psi}_H(\bar{1})\, \hat{\psi}_H^+(\bar{1}^+) \ldots\}$$

Where $\bar{1}^+$ has a time later on γ than $\bar{1}$.

Then we can write

$$\frac{1}{i^n}\, \mathcal{T}\left\{\hat{\psi}_H(1) \ldots \left(i \frac{d}{dz_k} \hat{\psi}_H(k)\right) \ldots \hat{\psi}_H(n)\, \hat{\psi}_H^+(n') \ldots \hat{\psi}_H^+(1')\right\}$$

$$= h(k)\, \hat{G}_n(1, \ldots, n; n', \ldots, 1')$$

$$\pm \frac{1}{i^n} \int d\bar{1}\; v(k;\bar{1})\, \mathcal{T}\left\{\hat{\psi}_H(1) \ldots \hat{\psi}_H(n) \boxed{\hat{\psi}_H(\bar{1})\, \hat{\psi}_H^+(\bar{1}^+)} \hat{\psi}_H^+(n') \ldots \hat{\psi}_H^+(1')\right\}$$

$$= h(k)\, \hat{G}_n(1, \ldots, n; n', \ldots, 1') \pm i \int d\bar{1}\; v(k;\bar{1})\, \hat{G}_{n+1}(1, \ldots, n, \bar{1}; 1', \ldots, n', \bar{1}^+)$$

and analogous for the 2nd equation with $\bar{z}_1^-$.

Inserting into the EOMs we find

$$\boxed{\begin{aligned}
\left[i \frac{d}{dz_k} - h(k)\right] \hat{G}_n(1, \ldots, n; 1', \ldots, n') &= \pm i \int d\bar{1}\; v(k;\bar{1})\, \hat{G}_{n+1}(1, \ldots, n, \bar{1}; 1', \ldots, n', \bar{1}^+) \\
&\quad + \sum_{j=1}^{n} (\pm)^{k+j}\, \delta(k, j')\, \hat{G}_{n-1}(1, \ldots \sqcap_k \ldots n; 1' \ldots \sqcap_{j'} \ldots n') \\[1em]
\hat{G}_n(1, \ldots, n; 1', \ldots, n') \left[-i \frac{\overleftarrow{d}}{dz_{k'}} - h(k')\right] &= \pm i \int d\bar{1}\; v(k';\bar{1})\, \hat{G}_{n+1}(1, \ldots, n, \bar{1}^-; 1', \ldots, n', \bar{1}) \\
&\quad + \sum_{j=1}^{n} (\pm)^{k+j}\, \delta(j; k')\, \hat{G}_{n-1}(1, \ldots \sqcap_j \ldots n; 1' \ldots \sqcap_{k'} \ldots n')
\end{aligned}}$$

- Hierarchy of operator equations in Fock space
- independent of specific shape of the contour
- basis for diagrammatic perturbation theory

end lecture