IV.3 Martin-Schwinger hierarchy

Now: step from correlator $\hat{G}$ to Green's function G

Def.: n-particle Green's function

$$G_n(1,\dots,n;1',\dots,n') \equiv \frac{\mathrm{Tr}\left[e^{-\beta\hat{H}^M}\hat{G}_n(1,\dots,n;1',\dots,n')\right]}{\mathrm{Tr}\left[e^{-\beta\hat{H}^M}\right]}$$

$$= \frac{1}{i^n}\frac{\mathrm{Tr}\left[\mathcal{T}\left\{e^{-i\int_\gamma d\bar{z}\hat{H}(\bar{z})}\hat{\psi}(1)\dots\hat{\psi}(n)\hat{\psi}^+(n')\dots\hat{\psi}^+(1')\right\}\right]}{\mathrm{Tr}\left[\mathcal{T}\left\{e^{-i\int_\gamma d\bar{z}\hat{H}(\bar{z})}\right\}\right]}.$$

For instance for G_1 with $z_1 < z_1'$ this means

$$e^{-\beta\hat{H}^M}\mathcal{T}\left\{\hat{\psi}_H(1)\hat{\psi}_H^+(1')\right\} = \pm\hat{U}(z_f,z_i)\hat{U}(z_i,z_1')\hat{\psi}^+(1')\hat{U}(z_1',z_1)\hat{U}(z_1,z_i)$$

$$= \pm\mathcal{T}\left\{e^{-i\int_\gamma d\bar{z}\hat{H}(\bar{z})}\hat{\psi}^+(1')\hat{\psi}(1)\right\}$$

$$= \mathcal{T}\left\{e^{-i\int_\gamma d\bar{z}\hat{H}(\bar{z})}\hat{\psi}(1)\hat{\psi}^+(1')\right\}$$

where we used $e^{-\beta\hat{H}^M} = \hat{U}(z_f,z_i)$ and field operators (anti-) commute under $\mathcal{T}$.

For example, G_1 at $z_1 = z$ and $z_1' = z^+ > z$ is proportional to the time-dependent ensemble average of $\hat{\psi}^+(x_1')\hat{\psi}(x_1)$ from which the time-dependent ensemble average of <u>any</u> one-body operator can be computed!

Same for n-body operators and G_n

For $z = t_0$: generalization to n-particle density matrices

The hierarchy equations for $\hat{G}_n$ derived in the previous chapter immediately implies by multiplication with the appropriate density matrix $\hat{\rho}$ and taking the trace:

$$\boxed{\begin{aligned}
&\left[i\frac{d}{dz_k} - h(k)\right] G_n(1,\dots,n;1',\dots,n') \\
&\quad = \pm i \int d\bar{1}\, v(k,\bar{1})\, G_{n+1}(1,\dots,n,\bar{1};1',\dots,n',\bar{1}^+) \\
&\qquad + \sum_{j=1}^{n} (\pm)^{k+j}\, \delta(k,j')\, G_{n-1}(1,\dots\overline{k}\dots n;1'\dots\overline{j'}\dots n') \\
&G_n(1,\dots,n;n',\dots,1')\left[-i\frac{\overleftarrow{d}}{dz_{k'}} - h(k')\right] \\
&\quad = \pm i \int d\bar{1}\, v(k',1)\, G_{n+1}(1,\dots,n,\bar{1};1',\dots,n',\bar{1}) \\
&\qquad + \sum_{j=1}^{n} (\pm)^{k+j}\, \delta(j,k')\, G_{n-1}(1,\dots\overline{j}\dots,n;1'\dots\overline{k'},\dots,n')
\end{aligned}}$$

Martin-Schwinger hierarchy

Examples of observables that are obtained from G_n's:

- Density $$n(x,z) = \frac{\mathrm{Tr}\left[e^{-\beta \hat{H}^M} \hat{\psi}_H^+(x,z)\, \hat{\psi}_H(x,z)\right]}{\mathrm{Tr}\left[e^{-\beta \hat{H}^M}\right]} = \pm i\, G_1(x,z;x,z^+)$$

- paramagnetic current density
$$\vec{j}(x,z) = \frac{1}{2mi} \frac{\mathrm{Tr}\left[e^{-\beta \hat{H}^M}\left(\hat{\psi}_H^+(x,z)\left(\vec{\nabla}\hat{\psi}_H(x,z)\right) - \left(\vec{\nabla}\hat{\psi}_H^+(x,z)\right)\hat{\psi}_H(x,z)\right)\right]}{\mathrm{Tr}\left[e^{-\beta \hat{H}^M}\right]}$$
$$= \pm \left(\frac{\vec{\nabla} - \vec{\nabla}'}{2m} G(x,z;x',z^+)\right)\Big/_{x'=x}$$

- interaction energy
$$E_{int}(z) = \frac{1}{2}\int dx \int dx'\, v(x,x';z) \frac{\mathrm{Tr}\left[e^{-\beta \hat{H}^M} \hat{\psi}_H^+(x,z)\, \hat{\psi}_H^+(x',z)\, \hat{\psi}_H(x',z)\, \hat{\psi}_H(x,z)\right]}{\mathrm{Tr}\left[e^{-\beta \hat{H}^M}\right]}$$
$$= -\frac{1}{2}\int dx \int dx'\, v(x,x';z)\, G_2(x',z,x,z;x',z^+,x,z^+)$$

<u>Important observation:</u> The derivation of the Martin-Schwinger hierarchy only depended on the behavior of operators under contour-time ordering $\mathcal{T}$ and the fulfillment of according Heisenberg equations of motion. The exact shape/type of contour was not used. The equations are therefore valid for <u>different closed contours</u> (see discussion later.)

Important: boundary condition, which follows from the definition:

$$\boxed{\begin{aligned} &G_n(1, \dots, (x_k, z_i), \dots, n; 1', \dots, n') = \pm \\ &\qquad G_n(1, \dots, (x_k, z_f), \dots, n; 1', \dots, n') \\ &G_n(1, \dots, n; 1', \dots, (x_k', z_i), \dots, n') = \pm \\ &\qquad G_n(1, \dots, n; 1', \dots, (x_k', z_f), \dots, n') \end{aligned}}$$

Example: 1-particle Green's function

$$G_1(x, z_i; x', z') = \pm G_1(x, z_f; x', z')$$
$$G_1(x, z; x', z_i) = \pm G_1(x, z; x', z_f)$$

Kubo-Martin-Schwinger relations

Kubo 1957

Martin & Schwinger 1959

Proof: Numerator in the definition is

$$\mathrm{Tr}\left[\mathcal{T}\left\{e^{-i\int_\gamma d\bar{z}\, H(\bar{z})}\, \hat{\psi}(1) \dots \hat{\psi}(n)\, \hat{\psi}^+(n') \dots \hat{\psi}^+(1')\right\}\right] = \dots$$

$$\hat{\psi}(k-1)\, \boxed{\hat{\psi}(x_k, z_f)}\, \hat{\psi}(k+1)$$

move outside $\mathcal{T}$ since z_f is latest time on γ

$\Rightarrow k-1$ (anti) commutations