IV. Many-body Perturbation theory (Stefanucci & van Leeuwen, Chapter ~~4~~ 5)

IV.1 Equations of motion on the contour

Def.: Contour evolution operator

"later on γ"

$$\hat{U}(z_2,z_1) = \begin{cases} \mathcal{T}\left\{e^{-i\int_{z_1}^{z_2} d\bar{z}\, \hat{H}(\bar{z})}\right\} & z_2 > z_1 \\ \bar{\mathcal{T}}\left\{e^{+i\int_{z_2}^{z_1} d\bar{z}\, \hat{H}(\bar{z})}\right\} & z_2 < z_1 \end{cases}$$

Properties: (1) $\hat{U}(z,z) = \mathbb{1}$

(2) $\hat{U}(z_3,z_2)\,\hat{U}(z_2,z_1) = \hat{U}(z_3,z_1)$

(3) Differential equation for $z > z_0$

$$i\frac{d}{dz}\hat{U}(z,z_0) = \mathcal{T}\left\{i\frac{d}{dz} e^{-i\int_{z_0}^{z} d\bar{z}\,\hat{H}(\bar{z})}\right\}$$

$$= \mathcal{T}\left\{\hat{H}(z)\, e^{-i\int_{z_0}^{z} d\bar{z}\,\hat{H}(\bar{z})}\right\} = \hat{H}(z)\hat{U}(z,z_0)$$

$$i\frac{d}{dz}\hat{U}(z_0,z) = -\hat{U}(z_0,z)\,\hat{H}(z)$$

Contour derivatives:

$\underline{z = t_-}$: $$\frac{d}{dz}\hat{A}(z) = \lim_{z'\to z}\frac{\hat{A}(z')-\hat{A}(z)}{z'-z} = \lim_{\varepsilon\to 0}\frac{\hat{A}_-(t+\varepsilon)-\hat{A}_-(t)}{\varepsilon} = \frac{d}{dt}\hat{A}_-(t)$$

$\underline{z = t_+}$: $$\frac{d}{dz}\hat{A}(z) = \frac{d}{dt}\hat{A}_+(t)$$ $\Rightarrow$ same derivative on γ_+ and γ_- if operators are the same

$\underline{z \in \gamma^M}$: $$\frac{d}{dz}\hat{A}(z) = \lim_{z'\to z}\frac{\hat{A}(z')-\hat{A}(z)}{z'-z} = \lim_{\varepsilon\to 0}\frac{\hat{A}(t_0-i(\tau+\varepsilon))-\hat{A}(t_0-i\tau)}{-i\varepsilon}$$

$$= i\frac{d}{d\tau}\hat{A}(t_0-i\tau).$$

In particular $$\boxed{\hat{U}(t_2,t_1) = \hat{U}(t_{2-},t_{1-}) = \hat{U}(t_{2+},t_{1+})}$$

We can rewrite the ensemble average (Chapter I)

$$O(z) = \frac{\mathrm{Tr}\left[\mathcal{T}\left\{e^{-i\int_\gamma d\bar{z}\,\hat{H}(\bar{z})}\,\hat{O}(z)\right\}\right]}{\mathrm{Tr}\left[\mathcal{T}\left\{e^{-i\int_\gamma d\bar{z}\,\hat{H}(\bar{z})}\right\}\right]},$$

using $\underbrace{z_i \equiv t_{0-}}_{\text{initial point on } \gamma}$ and $\underbrace{z_f \equiv t_0 - i\beta}_{\text{final point on } \gamma}$, as

$$O(z) = \frac{\mathrm{Tr}\left[\hat{U}(z_f, z)\,\hat{O}(z)\,\hat{U}(z, z_i)\right]}{\mathrm{Tr}\left[\hat{U}(z_f, z_i)\right]}$$

$$= \frac{\mathrm{Tr}\left[\hat{U}(z_f, z_i)\,\overbrace{\hat{U}(z_i, z)\,\hat{O}(z)\,\hat{U}(z, z_i)}\right]}{\mathrm{Tr}\left[\hat{U}(z_f, z_i)\right]}$$

Heisenberg picture on the contour:

$$\boxed{\hat{O}_H(z) \equiv \hat{U}(z_i, z)\,\hat{O}(z)\,\hat{U}(z, z_i)}$$

which gives $\hat{O}_H(t_+) = \hat{O}_H(t_-) = \hat{O}_H(t) =$ operator in standard Heisenberg picture.

Equation of motion:

$$i\frac{d}{dz}\hat{O}_H(z) = \hat{U}(z_i, z)\left[\hat{O}(z), \hat{H}(z)\right]\hat{U}(z, z_i) + i\frac{\partial}{\partial z}\hat{O}_H(z)$$

$$= \left[\hat{O}_H(z), \hat{H}_H(z)\right] + i\underbrace{\frac{\partial}{\partial z}\hat{O}_H(z)}_{\text{explicit } t\text{-dep.}}$$

For field operators $\hat{\psi}^+(x)$ and $\hat{\psi}(x)$ with (anti)commutation relations $\left[\hat{\psi}(x), \hat{\psi}^+(y)\right]_\pm = \delta(x-y)$ for fermions $(-)$ or bosons $(+)$, see Stefanucci & van Leeuwen, Chapter 1.4.

Here it is sufficient to know that we write a Hamiltonian generally as

$$\hat{H}^M = \underbrace{\int dx \int dx' \, \hat{\psi}^+(x) \langle x | \hat{h}^M | x' \rangle \hat{\psi}(x')}_{\hat{H}_0^M}$$

$$+ \underbrace{\frac{1}{2} \int dx \int dx' \, v^M(x,x') \hat{\psi}^+(x) \hat{\psi}^+(x') \hat{\psi}(x') \hat{\psi}(x)}_{\hat{H}_{int}^M}$$

for 2-body interactions.

Then the equations of motion on the contour become

$$[\hat{n}_H(x',z) \equiv \hat{\psi}^+_H(x',z) \hat{\psi}_H(x',z)]$$

$$i \frac{d}{dz} \hat{\psi}_H(x,z) = \int dx' \langle x | \hat{h}(z) | x' \rangle \hat{\psi}_H(x',z)$$

$$+ \int dx' \, v(x,x',z) \, \hat{n}_H(x',z) \, \hat{\psi}_H(x,z)$$

$$-i \frac{d}{dz} \hat{\psi}^+_H(x,z) = \int dx' \, \hat{\psi}^+_H(x',z) \langle x' | \hat{h}(z) | x \rangle$$

$$+ \int dx' \, v(x,x',z) \, \hat{\psi}^+_H(x,z) \, \hat{n}_H(x',z)$$

where $\hat{h}(z = t_\pm) = \hat{h}(t)$ $\quad v(x,x',t_\pm) = v(x,x',t)$

$\hat{h}(z \in \gamma^M) = \hat{h}^M$ $\quad v(x,x',z \in \gamma^M) = v^M(x,x')$