Topological Band Theory

References:
- Bernevig & Hughes, Topological Insulators and Superconductors
- Lecture Notes by C. Kane
- edX online course: Topology in condensed matter: tying quantum knots (Delft X)

Plan for the lecture:

1. 0D examples, topology, and symmetry
2. 1D example: Bulk-edge correspondence in Kitaev chain
3. Charge pumping
4. Quantum Hall Effect
5. Chern Insulators
6. From Quantum Spin Hall effect to topological insulators
7. Kubo formula and TKNN invariant
OD examples: topology and symmetry

Topology: Discrete things that cannot change continuously
Interesting if discreteness has measurable physical consequences. Topology can be used to classify physical systems with an excitation energy gap.

Zero-dimensional quantum systems

Consider a quantum system with $N$ states

\[ H |n\rangle = E_n |n\rangle \]

e.g., a small quantum dot

Fermi level $E_F$ in metallic lead

\[ \Rightarrow \text{states with } E_n < E_F \text{ are filled in dot} \]

(set $E_F = 0$)

Topology and gapped systems

Def.: two systems are topologically equivalent if they can be continuously deformed into each other without closing the energy gap.

Choose a random real symmetric $H$ and deform it into $H'$:

\[ H(\alpha) = \alpha H' + (1-\alpha)H \]

$\alpha = 0$: initial $H$, $\alpha = 1$: final $H'$
For example:

\[ E_1 \]

zero-energy crossing: breaks "gap condition".

Are \( H \) and \( H' \) topologically equivalent according to the above definition?

Other example:

\[ E_2 \]

two crossings - but maybe we can change the path between \( H \) and \( H' \) to push this level below zero. 

\( \Rightarrow \) need an easier way to figure out topological equivalence.

Concept: topological invariant

Idea: count \# levels below \( E = 0 \) \( \equiv \) \# filled states

\( \equiv \) topological invariant \( Q \)
Consider a Hamiltonian \( H \) with a symmetry constraint: 
If a unitary \( U \), e.g., \( U = \sigma_z \otimes I \), such that 
\[ U^\dagger H U = H. \]

\( H \) can be block-diagonalized due to a conservation law. 
\[ H_{4 \times 4} = \begin{bmatrix} \mathbb{I}_{2 \times 2} & \mathbb{0}_{2 \times 2} \\ \mathbb{0}_{2 \times 2} & \mathbb{0}_{2 \times 2} \end{bmatrix} \]

\( \Rightarrow \) Look at each subblock separately, \( \mathcal{Q}[H] = \mathcal{Q}[H_2] + \mathcal{Q}[H_1] \).

\( \Rightarrow \) Unitary symmetries are useful to reduce dimensionality, but otherwise boring.

Other symmetries are more interesting, e.g., time-reversal symmetry.
Time-reversal symmetry (trs)

Are real matrices special? Yes: they are manifestations of trs.

trs is represented by an anti-unitary operator that can be written as \( \mathcal{T} = U K \)

For real matrices, \( H = H^* \), and they commute with \( \mathcal{T} \) since complex conjugation does nothing.

For a random complex Hamiltonian, nothing changes when \( \mathcal{T} = K \) (U = 1) — Q changes as energy levels cross zero.

Important case where trs makes a difference: spin \( \frac{1}{2} \) systems. For these: \( \mathcal{T} = i \sigma_y K \), \( \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \)

\( \Rightarrow \mathcal{T}^2 = -1 \).

Then trs means \( H = \sigma_y H^* \sigma_y \)

\( \Rightarrow \) every energy eigenvalue is doubly degenerate (Kramers degeneracy)

\( \Rightarrow \) Q can only take even numbers

\( Q = 0, 2, 4, \ldots \)

\( \Rightarrow \) example for how discrete symmetries influence topology
Sublattice symmetry

Only non-zero matrix elements between sublattices A and B, which are degenerate.

\[ H = \begin{pmatrix} 0 & H_{AB} \\ H_{AB}^+ & 0 \end{pmatrix} \]

\( \Rightarrow \) Introduce diagonal matrix \( \sigma_z \) which is +1 for sublattice A and -1 for B:

\[-H = \sigma_z \cdot H \cdot \sigma_z \]

\( \Rightarrow \) If \( (\psi_A, \psi_B)^T \) is an eigenvector with energy \( E \), then \( (\psi_A, -\psi_B)^T \) is an eigenvector with energy \(-E\). \( \Rightarrow \) Symmetric spectrum due to sublattice symmetry

\( \Rightarrow \) The topological invariant \( \Omega \) never change for systems with sublattice symmetry

\( \Rightarrow \) Extra symmetry may render topological classification trivial!

Test: Which symmetry does not restrict the possible values of \( \Omega \) for quantum dots?

- A spinless trs    - B sublattice symmetry    - C spinful trs
Particle-hole symmetry

Example: Superconductors (SC)

SC has pairing terms

\[ H = \sum_{nm} H_{nm} C_n^+ C_m + \frac{1}{2} \Delta_{nm}^* C_n^+ C_m C_m^+ C_n^+ + \Delta_{nm} C_n C_m C_m^+ C_n^+ \]

\( C_n^+ \), \( C_n \) : fermionic creation/annihilation op's

\( C_n^+ C_m + C_m C_n^+ = \delta_{nm} \)

\( \Delta_{nm} \): antisymmetric matrix

\( H_{nm} \): dot Hamiltonian without SC pairing

\( H \) does not preserve particle number, but preserves its parity, i.e., whether # electrons is even or odd.

Using \( C = (c_1, c_2, ..., c_N, c_1^+, c_2^+, ..., c_N^+)^T \) we write

\[ H = \frac{1}{2} C^+ H_{BdG} C \]

with the Bogoliubov-de Gennes Hamiltonian

\[ H_{BdG} = \begin{pmatrix} H & \Delta \\ -\Delta^* & -H^* \end{pmatrix} = \begin{pmatrix} \text{electrons pairing} \\ \text{pairing holes} \end{pmatrix} \]

Since holes and electrons are related, \( H_{BdG} \) automatically has an extra symmetry exchanging electrons and holes: antiunitary \( \mathcal{P} = T_x K \), where \( T_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \) acts on the particle and hole blocks.
$P \ H_{06} \ F^{-1} = -H_{06}$

Particle-hole symmetry is represented by an antiunitary operator that anticommutes with the Hamiltonian. Because of the minus sign in front of $H_{06}$ for phs, the spectrum of $H_{06}$ must be symmetric around energy zero (Fermi level): for every eigenvector $\Psi = (u, v)^T$ of $H_{06}$ with energy $E$, there is a ph-symmetric eigenvector $P \Psi = (v^*, u^*)^T$ with energy $-E$.

So as for sublattice symmetry, phs forces $Q$ to remain the same; but, for $H_{06}$, zero-energy crossings can occur for phs — what happens there?

Fermion parity switches

For $H_{06}$ we had to double the number of states. The pairs at $\pm E$ do not correspond to different quantum states, but to a single quantum state.

Bogoliubov quasi-particle = coherent superposition of electrons & holes; energy $E$, operator $\alpha = uc^+ + vc$. Populating the particle state at $-E$ is the same as emptying the $+E$ state.
When a pair of Bogoliubov states crosses zero, the excitation energy \( E \) changes sign, and it becomes favorable to add a Bogoliubov quasiparticle to/ remove it from the dot. Level crossing \( \equiv \) changes in fermionic parity in dot ground state from even to odd or vice versa. \( \equiv \) fermion parity switches.

Hence preserves fermion parity if there are no Bogoliubov quasiparticles crossing zero energy: ground state fermion parity is the topological invariant of the system.

The topological invariant depends on the nature (symmetry) of the system under consideration!

Non-superconductors: \( Q = \# \) of negative energy eigenvalues

Superconductors: Level crossings can occur but \( \# \) of negative eigenvalues does not change, but fermion parity changes. \( Q \equiv \) parity is a suitable topological invariant.

Can we compute the parity directly from \( \tilde{H}_{\text{BdG}} \)?

The Pfaffian invariant

Basis transformation: \( \tilde{H}_{\text{BdG}} = \frac{1}{2} \left( \begin{array}{cc} 1 & 1 \\ i & -i \end{array} \right) H_{\text{BdG}} \left( \begin{array}{cc} 1 & -i \\ 1 & i \end{array} \right) \)

\( \Rightarrow \tilde{H}_{\text{BdG}} = \frac{1}{2} \left( \begin{array}{cc} H - H^* + \Delta - \Delta^* & -i \left( H - iH^* + i\Delta + i\Delta^* \right) \\ iH + iH^* + i\Delta + i\Delta^* & H - H^* - \Delta + \Delta^* \end{array} \right) \) antisymmetric
\( \Delta \) is antisymmetric. Since \( H \) is Hermitian, \( H - H^* \) is also antisymmetric and \( H + H^* \) is symmetric.

\[ \Rightarrow \quad \text{Adj} \quad \text{is antisymmetric.} \]

Pfaffian is defined for antisymmetric matrices.

Basic idea: eigenvalues of antisymmetric matrices come in pairs. For \( \text{Adj} \), these are \( \pm E_n \).

\[
\text{Determinant} = \prod_{n} (-E_n^2).
\]

The Pfaffian allows to take the \( \prod \) of the determinant: \( \pm i \prod En \), in such a way that the sign of the product is uniquely defined. At a fermion parity switch, a single \( E_n \) changes sign, such that the Pfaffian changes sign while the determinant remains the same.

\[
Q_{\text{Adj}} = \text{sign} \left[ \text{Pf} (i \cdot \text{Adj}) \right]
\]

\[
\text{Pf} (A) = \sqrt{\det (A)}.
\]

Test: What happens to the topological invariant when we take \( \Delta = 0 \) in \( \text{Adj} \)?

\( \Delta = 0 \) not allowed.

1. Pfaffian still captures all topological properties.
2. \( H \) loses \( g \)-symmetry and becomes topologically trivial.
3. \( H \) has a new conservation law.
4. Each \( A \) two blocks have their own invariant.
Summary of 1:

- Simplest topological invariant of zero-dim. systems:
  number of negative-energy states = matrix signature of Hamiltonian (zeroth Chern number)

- Conservation law (unitary) $\Rightarrow$ H block-diagonal $\Rightarrow$ study topology of individual blocks

- Number of filled states becomes even for spinful time-reversal symmetry

- Sublattice symmetry: number of filled state becomes constant

- Particle-hole (charge conjugation) symmetry: makes signature constant like sublattice symmetry but generates a new kind of invariant: the sign of the Pfaffian, which can only take two values: $\pm 1$ = parity of the electron number in the ground state

Symmetries and conservation laws define the type and existence of topological invariants.
Fermionic operators \( c, c^+ \):
\[
\begin{align*}
cc^+ + c^+ c &= 1 \\
-\hat{c}^2 &= 0 \\
-\hat{c}^{+2} &= 0
\end{align*}
\]

Two states \( |0\rangle, |1\rangle \) with \( c^+ |0\rangle = |1\rangle \), \( c |0\rangle = 0 \)
\[
\begin{array}{r}
|0\rangle \\
\hline
1 \delta
\end{array}
\]

Rewrite fermions using Majorana operators \( \gamma_1, \gamma_2 \):
\[
\begin{align*}
c^+ &= \frac{1}{2} (\gamma_1 + i\gamma_2) \\
c &= \frac{1}{2} (\gamma_1 - i\gamma_2)
\end{align*}
\]

Inverse transformation:
\[
\begin{align*}
\gamma_1 &= c + c^+ \\
\gamma_2 &= i (c - c^+)
\end{align*}
\]
\[
\boxed{
\begin{align*}
c^+ c &= \gamma_1^+ \gamma_1 = 1 \\
c - c^+ &= \gamma_2^+ \gamma_2 = 0
\end{align*}
\]

\[=\] there is no "occupancy number operator" for Majoranas!

For normal (Dirac) fermions, \( c \neq c^+ \), and \( \hat{n} = c^+ c \)
counts the occupation of a fermionic state:
\[
\begin{align*}
\langle 0 | \hat{n} | 0 \rangle &= 0 \\
\langle 1 | \hat{n} | 1 \rangle &= 1
\end{align*}
\]

For Majoranas:
\[
\gamma_1^+ \gamma_1 \neq \gamma_1^2 = 1 \rightarrow \text{operator identity}
\]

It means that \( \langle 4 | \gamma_1^+ \gamma_1 | 4 \rangle = 1 \) independent of \( |4\rangle \).
And we have $\gamma_1 \delta_2 + \gamma_2 \delta_1 = 0$ — Majoranas for different modes anticommute. That is why we call them "fermions".

Since $c^+ = \frac{1}{2}(\gamma_1 + i \gamma_2)$ we can view two

complex  real  real  real

Majoranas as the real and imaginary parts of one fermion.

Majoranas must come in pairs.

Can two Majoranas be separated?

**Domino model**

\[
\begin{array}{cccc}
\gamma_1 & \delta_1 & \gamma_2 & \delta_2 \\
0 & 0 & 0 & 0 \\
\hline
n=1 & n=2 & n=3 & n=4
\end{array}
\]

\[H_\mu = -\mu N = -\mu \sum_{n=1}^{N} c_n^+ c_n \triangleq \text{fermion filling}\]

\[= \frac{i}{2} \mu \sum_{n=1}^{N} \delta_{2n-1} \delta_{2n} \triangleq \text{Majorana pairing}\]

But we would like a pairing like so:  

\[\Rightarrow \text{excitons gapped}\]

\[\Rightarrow \text{unpaired Majoranas}\]

\[\Rightarrow H_\mu = i \mu \sum_{n=1}^{N} \delta_{2n} \delta_{2n+1} \triangleq \text{fermion hopping}\]

\[\Rightarrow \delta_1 \text{ and } \delta_{2N} \text{ do not appear in } H_\mu\]

\[/13\]
\( \Rightarrow \) \( \mathcal{H}_t \) has two zero-energy states localized at the edges

\( \Rightarrow \) all other states have energy \( \pm 1/t \), independently of chain length

\( \Rightarrow \) 1D system with gapped bulk and zero-energy edge states

**Kitaev chain model**

\[ \gamma_{2n-1} = (c_{n+}^+ + c_n^-) \]

\[ \gamma_{2n} = -i (c_{n+}^+ - c_n^-) \]

*tight-binding model for 1D superconductor (SC) wire:*

\[ \mathcal{H} = -\mu n \sum_n c_n^+ c_n - t \sum_n (c_{n+}^+ c_n^+ + h.c.) + \Delta \sum_n (c_{n+}^+ c_{n+ + h.c.}) \]

3 real parameters: chemical potential \( \mu \)

hopping integral \( t \)

SC pairing energy \( \Delta \)

*Pairing terms in Majumdar:

\[ \Delta c_n c_n^+ + h.c. = \frac{\Delta}{4} \left( \gamma_{2n+1} \gamma_{2n+2} - i \gamma_{2n} \gamma_{2n+1} - i \gamma_{2n+1} \gamma_{2n+2} \right) + \frac{\Delta}{4} \left( \gamma_{2n+1} \gamma_{2n+2} - i \gamma_{2n} \gamma_{2n+1} + i \gamma_{2n+1} \gamma_{2n+2} \right) \]

\[ = \frac{i\Delta}{2} \left( \gamma_{2n+1} \gamma_{2n} + \gamma_{2n+2} \gamma_{2n-1} \right) \]

*Hopping:

\[ -t c_{n+}^+ c_n^+ + h.c. = -\frac{t}{4} \left( \gamma_{2n+1} \gamma_{2n+2} + \gamma_{2n} \gamma_{2n+1} + i \gamma_{2n+1} \gamma_{2n+2} - i \gamma_{2n} \gamma_{2n+1} \right) \]

\[ -\frac{t}{4} \left( \gamma_{2n+1} \gamma_{2n+2} + \gamma_{2n} \gamma_{2n+1} - i \gamma_{2n+1} \gamma_{2n+2} + i \gamma_{2n} \gamma_{2n+1} \right) \]

\[ = \frac{it}{2} \left( \gamma_{2n+1} \gamma_{2n} - \gamma_{2n+2} \gamma_{2n-1} \right) \]
\( \Rightarrow \) if \( \Delta = t \), the \( \delta_{2n,2n-1} \) terms vanish.

if \( \mu = 0 \) in addition, \( H = \ldots \)

\( \Rightarrow \) \( \Delta = t \neq 0 \), \( \mu = 0 \) creates exactly the unpaired Majorana edge mode model

trivial case (fully paired): \( \mu \neq 0, \Delta = t = 0 \)

\( \Rightarrow \) there must be a phase transition in between!

Remember: It is useful to write a superconducting Hamiltonian in Bogoliubov-de-Gennes form:

\[
H = \frac{1}{2} C^+ H_{\text{BdG}} C
\]

\[
C \equiv (c_1, \ldots, c_N, c_1^+, \ldots, c_N^+)^T
\]

\( 2N \times 2N \) matrix \( H_{\text{BdG}} \) can be written using Pauli matrices in particle-hole space, and \( |n\rangle \equiv (0, \ldots, \delta_{n,N}, 0)^T \)

a column vector corresponding to the \( n \)-th site of the chain. For example: \( C^+ T_x \langle n | C = 2 c_n^+ c_{n+1} \).

\[
H_{\text{BdG}} = - \sum_n \mu T_x |n\rangle \langle n| - \sum_n \left[ (t T_x + i \Delta T_y) |n\rangle \langle n+1| \right. \\
\left. + \text{h.c.} \right]
\]

\( H_{\text{BdG}} \) acts on states in a basis \( |n\rangle |\uparrow \rangle \)

with \( J = \pm 1 \) corresponding to electron and hole states.

\( H_{\text{BdG}} \) has p.h.s.: \( P H_{\text{BdG}} P^{-1} = -H_{\text{BdG}}, P = T_x K \)

complex conj. / 15
Topological protection of Majorana edge modes

Is fine-tuning ($\mu=0$) necessary to have unpaired Majoranas? What happens when $\mu \neq 0$?

$\rightarrow$ demo $N=25$ chain

Majoranas persist until $\mu \approx 2t$, where the bulk gap closes.

$\Rightarrow$ Majoranas only merge with the higher-energy states in the bulk, originally at $2t$, come close to zero energy. **No fine-tuning required!**

Majorana edge modes are protected by the bulk energy gap, and by particle-hole symmetry!
Only way to move Majorana zero modes (M2M) away from zero is 
to couple them to each other! This is impossible as long as they are spatially separated and the bulk gap persists.

\[ \Rightarrow \] one needs to close the gap to destroy M2M
\[ (\mu = 2t : \text{gap closes}) \]

**Hint from this:** Information about the edge modes is already contained in the bulk states!

There is a deep reason: bulk-edge correspondence

Test: What happens if we remove a single Majorana site in the topological phase?

- **A** We get a chain with a single unpaired M2M
- **B** Impossible; Electrons (pairs of Majoranas) are the only physical degree of freedom in the outside world
- **C** H becomes topologically trivial
- **D** Removing a single Majorana site is forbidden by particle–hole symmetry
Topological phases from the bulk

Momentum space (band structure):

Close the chain with periodic boundary conditions (PBC)

$$|k\rangle = (N)^{-1/2} \sum_{n=0}^{N} e^{-i k n} |n\rangle$$

PBC: $$\langle k|n=0\rangle = \langle k|n=N\rangle$$

$$\Rightarrow k$$ good quantum number, takes values $$\frac{2\pi}{N} p$$,

$$p = 0, \ldots, N-1$$

$$N \rightarrow \infty: \quad k \in [-\pi, \pi] = 1$$st Brillouin zone

$$H_{0dc} = \sum_{k} H(k) |k\rangle \langle k|$$

$$H(k) = \langle k|H_{0dc}|k\rangle = (-2t \cos k - \mu) T_x + 2 \Delta \sin k T_y$$

$$N \rightarrow \infty: \quad \sum_{k} \rightarrow \int \frac{dk}{2\pi}$$

Particle-hole symmetry in momentum space:

Careful with anti-unitary operators under basis transformations
because of the action of $$K$$ on the $$e^{-i k n}$$:

$$K e^{-i k n} = e^{i k n} K$$

Here: $$P |k\rangle |\uparrow\rangle = (\sum_{n} e^{-i k n})^{*} |n\rangle T_{x} |\uparrow\rangle^{*} = |k\rangle T_{x} |\uparrow\rangle^{*}$$

$$\Rightarrow P$$ changes $$k$$ to $$-k$$!

$$P H_{0dc} P^{-1} = \sum_{k} T_{x} H^{*}(k) T_{x} |k\rangle \langle k| -k\rangle \langle -k| = \sum_{k} \uparrow \downarrow T_{x} H^{*}(k) T_{x} |k\rangle \langle k|$$

momenta come in pairs $$(-k, k)$$ + symmetric points $$k = \frac{\pi}{N}$$
Particle-hole symmetry \( \mathcal{P} H_{\text{adc}} \mathcal{P}^{-\gamma} = -H_{\text{adc}} \) \( \Rightarrow \)

\[
H(k) = -J_x H^*(-k) J_x
\]

This is true because

\[
J_x H^*(-k) J_x = (+2t \cos k + \mu) T_x - 2\Delta \sin k T_y
\]

\[
= -H(k)
\]

particle-hole symmetry in \( k \) space

\( \Rightarrow \) given a solution with \( E \) and \( k \), there is also a solution with \( -E \) and \( -k \)

Band structure:

Diagonalize \( H(k) \)

\[
H(k) = \begin{pmatrix}
-2t \cos k - \mu & -2i \Delta \sin k \\
2i \Delta \sin k & +2t \cos k + \mu
\end{pmatrix}
\]

\( \Rightarrow \) \( E(k) = \pm \sqrt{(2t \cos k + \mu)^2 + 4\Delta^2 \sin^2 k} \)

\( \Rightarrow \) \( DEMO \)

\( \Rightarrow \) gapped at all \( k \) for \( \mu = 0 \) \( \) [why? before we had probing zero modes at \( \mu = 0 \)]

\( \Rightarrow \) bulk gap closes at \( \mu = \pm 2t \)

+ : band touching at \( k = \pi \)

- : band touching at \( k = 0 \)
Is there a phase transition? First: write down a simple effective model near the transition.

**Effective Dirac model:**

Focus $\mu = -2t$ at $k = 0$.

Linear Taylor expansion:

$$H(k) = m T_k + 2 \Delta k T_y$$

$$m \equiv -\mu - 2t \text{ mass term}$$

$$E(k) = \pm \sqrt{m^2 + 4 \Delta^2 k^2}$$

**Mass term is crucial:** It determines the gap and

- $m < 0$ for $\mu > -2t$: topological phase
- $m > 0$ for $\mu < -2t$: trivial phase

**Mass term changes sign at $\mu = -2t$!**

$m = 0$: $H(k)$ has two eigenstates $E = \pm 2\Delta k$, which are eigenstates of $T_y$.

$\Rightarrow$ equal-weight superpositions of electron + hole

$\Rightarrow$ these are Majorana modes

$\text{Velocity } v = \frac{E}{E} = \pm 2\Delta \text{ "right mover"}$

$E = +2\Delta k, L \to k$

$\text{Velocity } v = \frac{E}{E} = -2\Delta \text{ "left mover"}$

$k = 0, E = -2\Delta k$
For $m=0$, there are two branches of free-moving Majorana fermions!

Majorana modes at domain walls

What if $m$ changes sign in real space?

$$H(x) = -V T_j i \partial_x + m(x) T_2$$

Choose $m(x) \to \pm m$ for $x \to \pm \infty$, $m(x=0) = 0$

Look for $M^2 \Psi$: $H \Psi = 0$

$$\Rightarrow \partial_x \Psi(x) = \frac{1}{V} m(x) T_x \Psi(x)$$

$$\Rightarrow \Psi(x) = \exp \left( T_x \int_0^x \frac{m(x')}{V} \, dx' \right) \Psi(0)$$

$$\Rightarrow$$ two linearly independent solutions via eigenstates of $T_x$:

$$\Psi(x) = \exp \left( \pm \int_0^x \frac{m(x')}{V} \, dx' \right) \begin{pmatrix} 1 \\ \pm 1 \end{pmatrix}$$

$$= \text{only one of them is normalizable (the other is divergent)}$$
Bound state localized near the domain wall thanks to sign change of $n(x)$ — otherwise no normalizable M2M would exist.

Physics: $x < 0$: topological phase  
$x > 0$: trivial phase

At the interface there must be a zero mode!

Bulk topological invariant:

Effective model: sign of mass term helped understand topology. What can we use as topological invariant for a more general $H(C)$?

Hennius' derivation:
1. BdG Hamiltonian: for quantum dots we used the sign of the Pfaffian as topological invariant in the presence of particle-hole symmetry; the Pfaffian changes sign at every gap closing. Link m to a Pfaffian?

2. $H_{\text{BdG}}$: large matrix with phs.
   $\Rightarrow$ put $H_{\text{BdG}}$ in anti-symmetric form and compute its Pfaffian?

3. Pfaffian sign only changes when an eigenvalue passes through zero, phs $\Rightarrow E(k)$ comes with $-E(-k)$. Only two exceptions:
   $k=0$ and $k=\pi$ are mapped onto themselves for $k\rightarrow -k$.

   phs:
   \[ T_k H^*(0) T_k = -H(0) \]
   \[ T_k H^*(\pi) T_k = -H(\pi) \]

   $\Rightarrow H(0)$ and $H(\pi)$ can be anti-symmetrized individually.

4. gap closings happen exactly at $k=0$ and $k=\pi$, too.

   $\Rightarrow$ focus on $0$ and $\pi$. !
Anti-symmetrization (cf. Chapter 1):

\[
\tilde{H}(0) = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \begin{pmatrix} -2t & \mu \\ 0 & 2\mu \end{pmatrix} \begin{pmatrix} 1 & -i \\ i & 0 \end{pmatrix} = -i \begin{pmatrix} 0 & -2t - \mu \\ 2t + \mu & 0 \end{pmatrix}
\]

\[
\tilde{H}(\pi) = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \begin{pmatrix} \pi - 2t & \mu \\ 0 & 2\mu \end{pmatrix} \begin{pmatrix} 1 & -i \\ i & 0 \end{pmatrix} = -i \begin{pmatrix} 0 & 2t - \mu \\ -2t + \mu & 0 \end{pmatrix}
\]

\[\implies \text{Pf} \left[ iH(0) \right] = -2t - \mu \to \text{sign change at } \mu = -2t
\]

\[\text{Pf} \left[ iH(\pi) \right] = 2t - \mu \to -\quad -\quad \mu = +2t
\]

Topological invariant: \( \mathcal{Q} = \text{sign} \left( \text{Pf} \left[ iH(0) \right] \text{Pf} \left[ iH(\pi) \right] \right) \)

\( \mathcal{Q} = -1: \) bulk is in topological phase

\( \mathcal{Q} = +1: \) trivial "

The topological invariant \( \mathcal{Q} \) cannot change under continuous deformations of the Hamiltonian unless the gap closes.

Connection between bulk invariant and edge modes

Physical meaning of \( \mathcal{Q} \)?

Sign Pfaffian of BdG Hamiltonian: ground state fermion parity

Product of \( \text{Pf} \left[ iH(0) \right] \text{Pf} \left[ iH(\pi) \right] \): we are comparing the fermion parities of the \( k=0 \) and \( k=\pi \) states:

\( \mathcal{Q} = -1 \) only if they are different.
if we continuously deform $H(0)$ into $H(\pi)$ without breaking pbs, we must encounter a zero-energy level crossing (fermion parity switch).

Implementation: Change from PBC to anti-periodic BC

$$<k| n=0> = - <k| n=N>$$

$\Rightarrow$ allowed momenta $k = \frac{2\pi}{N} p + \frac{\pi}{N}$ APBC $p=0, \ldots, N-1$

$\Rightarrow$ what is the parity difference between the PBC chain and the APBC chain?

$k=0$ is always present in PBC chain. $p=0, \ldots, N-1$

$k=\pi$ is present for $N$ even in PBC $N$ odd in APBC

$\Rightarrow$ in either case, the difference in ground-state fermion parities between the PBC and APBC chains is equal to $\chi$!

Check: Consider Kane ring with $\Delta = t$. Change of hopping from $t$ to $-t$ on the last bond between $n=N-1$ and $n=0$ going from PBC to APBC chain. This can be done continuously w/o breaking pbs by setting the last hopping equal to $t(1-2i)$ with $i\in (0,1)$.
\[ \Rightarrow \text{Demo energy spectrum } E(\mu) \text{ as } \mu \text{ varies through the gap closing} \]

\[ \mu < 2t : \ E(\mu) \text{ crosses } 0 \text{ at } \lambda = \frac{1}{4} \]

\[ \Rightarrow \text{different parity at } \lambda = 0 \text{ and } \lambda = 1 \]

\[ \lambda = \frac{1}{4} : \text{ cut in chain (} t = 0 \text{ at both ends)} \]

\[ \Rightarrow \text{two MFLs in topological phase!} \]

\[ \mu > 2t : \text{ no level crossing through } 0 \]

\[ \Rightarrow \text{same groundstate fermion parity at } \lambda = 0 \text{ and } \lambda = 1 \]

\[ \lambda = \frac{1}{4} : \text{ no MFLs } \Rightarrow \text{consistent with trivial open chain} \]

Essence of bulk-boundary correspondence: \( Q = -1 \)

nontrivial bulk invariant for closed chain implies existence of unpaired Majoranas in the open chain.

Correction of \( Q \) to measurable quantity: groundstate fermion parity
Test: What happens when we take a 100-site topological Kitaev chain and change $\mu$ to a very large negative (trivial) value for the last 50 sites? (select all that match)

- Gap at last 50 sites closes, then reopens
- Majoranas get destroyed
- One of the Majoranas moves from the end to the middle of the chain
3. Charge pumping

Thouless pumps and winding invariant

Hamiltonians with parameters:

Consider a Hamiltonian that is periodic in a parameter:

\[ H(t + T) = H(t) \quad \text{for period } T \]

Examples:
- band structure: \( H(k) \) has period \( 2\pi \) (in 1D)
- time-dependent Hamiltonian under periodic driving

We want to study \( H(t) \) with slow changes in \( t \), such that the system remains in the ground state (gapped system). Slow: adiabatic!

Note: \( T = \infty \) is possible; \( H(-\infty) = H(+\infty) \).

Important: parameter space must be compact to define topological invariants.

Quantum pumps:

Consider a 1D region coupled to two electrodes with a sine-shaped confining potential.

\[ U(x) \]
\[ H(t) = \frac{\hbar^2}{2m} + A \left[ 1 - \cos \left( \frac{\pi}{\lambda} + 2\pi \frac{t}{T} \right) \right] \]

for \( x \) in the central region.

Choose \( A \gg \frac{1}{\hbar^2} \) \( \rightarrow \) strong potential

\( \frac{\hbar^2}{2m} \ll A \rightarrow \) states bound in minima have small overlap

Near minima: periodic potential \( \rightarrow E_n = (n + \frac{1}{2}) \hbar \omega_c \)

\[ \omega_c = \sqrt{\frac{A}{\hbar^2}} \]

Each minimum: integer number of electrons depending on chemical potential \( \mu \)

One time period: an integer number \( N \) of electrons is pumped from the left to the right electrode

**Quantitation of pumped charge**

Is the integerness of pumped charge a topological effect, or is it just due to finetuning (because the wells are deep) ?

Thought experiment: "dry out" pump by emptying the middle region pushing \( n_L \) electrons to the left \( n_R \) electrons to the right

1. emptying middle region: sides disconnected \( \rightarrow \) number of electrons must be integer on either side
2. adiabaticity: system always in an eigenstate
3. drying out → pump integer number of charges
4. adiabatic manipulation only possible if $H$ is always gapped

$\rightarrow$ the number of electrons $N$ pumped per cycle is integer as long as the bulk of the pump remains gapped.

$\Rightarrow$ $N$: topological invariant

In fact, $N$ is a Chern number (TKNN invariant), which will be introduced later.
4) Quantum Hall Effect (QHE)

Laughlin argument for quantization:

Classical Hall Effect:

\[
\mathbf{F}_{\text{Lorentz}} \quad \mathbf{B} \quad \mathbf{I} \quad \mathbf{F}_{\text{Lorentz}}
\]

\[\Rightarrow \text{Hall voltage develops to counteract } \left| \mathbf{F}_{\text{Lorentz}} \right| = F_L \]

\[V_H \]

Since \( F_L \sim B \Rightarrow V_H \sim B \)

\[\rho_{xy} \sim V_H \]

\[\rho_{xx} \]

but: at larger \( B \) there are plateaux!

\( \rho_{xy} \) (the Hall resistivity) DOES NOT CHANGE and

is quantized in inverse integers of \( \frac{\hbar}{e^2} \)

\[\rho_{xy} = \frac{1}{4} \frac{\hbar}{e^2} , \quad \frac{\hbar}{e^2} = 25,812,807 \sim \]

\( = R_K \), von Klitzing

constant

\[3/9\]

\[ \rightarrow \text{integer quantum Hall effect (IQHE).} \]

Thouless, Kohmoto, Nightingale, den Nijs (TKNN):

IQHE is a topological phenomenon related to a topological invariant (Chern number).

At the same time: **longitudinal resistivity** \( \rho_{xx} \) vanishes.

Reason: current and electric field are \( \perp \) to each other.

Curiously, this implies that the **longitudinal conductivity** \( \sigma_{xx} \) also vanishes.

Hall conductivity is quantized: \( \sigma_{xy} = \frac{e^2}{h} \).

At higher temperature: \( \rho_{xx} \) and \( \sigma_{xx} \) show activated behavior \( \sim e^{-T_0/T} \). → system is gap in the bulk.

**Twist:** Gap must close at the sample edges.

Consider a cylinder of IQHE material (radius \( R \)).
flux $\rightarrow$ azimuthal electric field $\rightarrow$ current $\rightarrow$ polarization of charge $Q$ between the edges.
Charge cannot relax because $\sigma_{xx} = 0$!
$\rightarrow$ charge accumulates — it needs states to do this!
Since the perturbation (magnetic field) is slow (low-frequency) and small (just one flux quantum), it is low-energy and scales like $1$/spatial size.
$\rightarrow$ these excited states must be close to the chemical potential.
$\rightarrow$ the gap must close at the edge!

At the interface between two regions with different Hall conductivities, there must be a gap closure.

3 pieces:  • left edge ('1D conductor')
            • interior ('2D insulator')
            • right edge ('1D conductor')

But: Edges are not like 1D wires because charge in them is not conserved! If an edge is peeled off, the smaller cylinder will again have gapped edge states.

Quantum mechanics: spectrum of cylinder must be periodic in the flux with periodicity $\frac{\hbar}{e} = \Phi_0$ (flux quantum)
$\rightarrow$ Aharonov–Bohm effect.
Start with ground state of cylinder at $\Phi = 0$.

Change flux to $\Phi = \Phi_0 \to$ system still in eigenstate.

Not ground state because charge has flown.

For non-interacting electrons: eigenstates = products of single-particle states; occupied + unoccupied.

Ground state: fill states below chemical potential.

After one flux quantum: again eigenstate, some filled states are empty, some empty ones are now filled.

Only available states close to $g_0$ are at edges:

An integer number of electrons gets pumped between the edges!

$\Rightarrow$ Hall conductivity is quantized in units of $e^2/h$

This is again a Thouless charge pump.

Underlying physics: electrons in Landau levels, cyclotron orbits. Role of disorder: In clean sample, Landau levels get filled in continuously; with disorder, there are localized states in between Landau levels; when these are filled, the plateaus appear.

Important ingredient to QHE: time-reversal symmetry breaking! Here: magnetic field. But there are other cases, too.
Chiral edge states:

We have seen that the charge-pumping argument implies conducting edge states. Intuitive quasi-classical picture:

- Cyclotron orbits due to Lorentz force
- Right-moving skipping orbits
- Left-moving skipping orbits

$\Rightarrow$ Chiral edge states: right and left movers on each edge

$\Rightarrow$ Chirality due to $\Theta$-breaking determined by direction of $B$-field

$\Rightarrow$ Another instance of bulk-boundary correspondence
Chern Insulators

Strategy to construct a lattice model for the QHE:

Idea: Kitaev chain model helped us understand Majoranas. QHE: One way to get QHE is to place electrons in a magnetic field. Now we would like to find a simple tight-binding model for the QHE which also allows us to get rid of the magnetic field.

Such models are called Chern insulators (CI).

QHE without magnetic field: Quantum Anomalous Hall Effect (QAHE)

First CI model: Haldane 1988 - dressed graphene lattice

Now: more natural approach - dominoes again?

Two key aspects:

1. QHE edge cannot exist in isolation from bulk, relies on bulk-boundary correspondence ("chiral anomaly").
   - Kitaev chain: MBL
   - QAHE: chiral edge states

2. Find a lower-dimensional building block
   - Kitaev chain: fermionic sites = pairs of Majoranas
First: find 1D system that can host pair of chiral edge states
- left + right movers, on top of each other
- put 1D lego piece on top of each other and couple them like we did for the Kitaev chain \( \rightarrow \) 2D system with partially separated chiral edge states = QAHE!

Plan: \[ Y \uparrow \]

1D wire with pair of edge states:

Reminder, Dirac model \( H = \Delta k_y T_y \) of the Kitaev chain at the critical point (topological phase transition).

Kitaev model:

Critical point \( \mu = -2t \):

Near \( k = 0 \) : pair of states with wave functions related to eigenvalues \( \pm 1 \) of \( T_y \), and with opposite velocities.
"Problem": Kitaev model is superconducting with phs, and T matrices refer to particles and holes. — should not cite for the QHE

Solution: re-interpret T matrices as acting on the space of left and right movers.

Generic strategy: phase transition points of lower-dim. models are good starting points to construct higher-dim. topological models.

Test: quantum Hall effect and Kitaev chain can have chiral states. What is the fundamental difference apart from dimensionality?

A) QHE edge states go in opposite directions while Kitaev states go in the same.

B) QHE edge states go in same direction while Kitaev states go in opposite directions.

C) QHE edge states always cross zero at k=0 while Kitaev states don’t.

D) Kitaev chiral states only exist at specific parameter values while QHE edge states are more robust.
Model Hamiltonian:

- stack x-oriented chains together along y.
- chain index \( n_y \)
- replace \( k \rightarrow k_x \)
- single chain: \( \left[ - (2t \cos k_x + \mu) T_z + \Delta \sin k_x T_y \right] \otimes |n_y \rangle \langle n_y | \) (projector onto chain \( n_y \))

- couple \( T_y = -1 \) bond of one chain to \( T_y = +1 \) bond of a neighboring chain \( \rightarrow \) QHE

- write this as \( \Psi \left| n_y \right\rangle \langle n_y + 1 \right| \otimes \left( T_z + i T_x \right) \)

  \[ T_z = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \]

  \[ |\Psi_{RL} \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} i \\ 1 \end{pmatrix} \quad \text{for } T_y |\Psi_{RL} \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} -i \\ -i \end{pmatrix} = \pm |\Psi_{RL} \rangle \]

  \[ (T_z + i T_x) |\Psi_{RL} \rangle = \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} i \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} i + i \\ i + i \end{pmatrix} = \begin{cases} 0 & \text{for } R \\ -2 |\Psi_{RL} \rangle & \text{for } L \end{cases} \]

- complete \( H \): sum over \( n_y \)
\[ H = \sum_{n_y} \sum_{k_x} \left\{ \left[ - (2t \cos k_x + \mu) T_z + \Delta \sin k_x T_y \right] \otimes |n_y \times n_y| \right\} - \gamma |n_y \rangle \langle n_y + 1| \otimes (T_z + i T_x) + \text{h.c.} \right\} \]

\[ \equiv \sum_{k_x} H(k_x) \]

should suffice to produce a quantum Hall state!

(need to keep in mind to use \( \mu = -2t \) for criticality of chains, but keep \( \mu \) free for now)

**Gap and edge states:**

Finite # chains \( n_y = 1, \ldots, N \).

Tune \( \mu = -2t \) : check that at \( k_x = 0 \) \( H \) has one right-moving edge state for \( n_y = N \) with eigenvalue \( \Delta k_x \), and one left-moving edge state for \( n_y = 1 \) with eigenvalue \( -\Delta k_x \).

**How:** Write \( H \) as matrix \( (N \times N) \) in \( |n_y \rangle \) basis

for \( 2t \cos k_x + \mu \approx 0 \), \( \sin k_x \approx k_x \):

\[ H(k_x \approx 0) = \begin{bmatrix}
\Delta k_x T_y & -\gamma(T_z+iT_x) \\
-\gamma(T_z-iT_x) & \Delta k_x T_y & -\gamma(T_z+iT_x) \\
& & \Delta k_x T_y & -\gamma(T_z+iT_x) \\
& & & \Delta k_x T_y & -\gamma(T_z+iT_x) \\
& & & & \Delta k_x T_y \\
\end{bmatrix} \]

**e.g.** \( N=4 \)

Exercise: find eigenstates and eigenvalues for \( N=2 \) and identify the edge states
Next: Are these edge states the only low-energy eigenstates — is the bulk gapped?

Exercise: Infinite stack = bulk only

→ go to momentum space along \( y \) also!

→ find \( H(k_x, k_y) \), plot the dispersion \( E(k_x, k_y) \)

Solution: 2D Bloch Hamiltonian for \( t = 1.0, \Delta = 0.7, \gamma = -0.5, \mu \in [-2, 0] \)

\[
H(k_x, k_y) = \left[ -(2t \cos k_x + \mu)T_x + \Delta \sin k_x T_y \right] \\
- 2\gamma \left[ \cos k_y T_x + \sin k_y T_x \right]
\]

\[
E(k_x, k_y) = \pm \sqrt{\Delta^2 \sin^2 k_x + \left( 2\gamma \cos k_y + \mu + 2t \cos k_x \right)^2 + 4\gamma^2 \sin^2 k_y}
\]

→ gapped except at \( \mu = -2t - 2\gamma \)

Finite ribbon: Plot all eigenvalues \( E(k_x) \) and check if there are edge states (Dirac-like crossing at \( k_x = 0 \))!

Details such as bulk spectrum and edge dispersion are different from the QHE with magnetic field; but bulk-edge correspondence tells us that our edge states are as robust as those of the QHE!

Test: How does our lattice model differ from the original quantum Hall effect?

\( \Delta \) Since there is no magnetic field, the lattice QHE model preserves time-reversal symmetry.
QHE in magnetic field has Landau levels that do not disperse in k (flat bands) while the bulk states disperse in the lattice

QHE in lattice has no chiral edge states, which arise from splitting orbit in magnetic field

in a magnetic field the filling fraction is fixed to integer per flux quantum, while in the lattice the filling fraction per unit cell is arbitrary.

Dirac equation at the phase transition:

Effective bulk Hamiltonian near transition \(0 = -\Delta k_x T_y - 2 \gamma k_y T_x + M T_z\)

\[ H_{\text{Dirac}}(k) = \Delta k_x T_y + 2 \gamma k_y T_x + M T_z \]

transition: \( m = 0 \): massless Dirac model

\( m > 0 \): topological phase

\( m < 0 \): trivial phase

Remember Kitaev chain: \( m(x) \) with sign change of \( m \)

\( \rightarrow \) domain wall

\( \rightarrow \) nondegenerate zero-mode

Here: fix \( k_y = 0 \) \( \rightarrow \) identical Hamiltonian as in C
zero mode: eigenstate of \( T_x \) with eigenvalue \( +1 \)

But here: zero mode not stationary \( E(k_y) \approx 2 \gamma k_y \)

\( \rightarrow \) velocity \( -2 \gamma \) at \( k_y = 0 \)
Summary:
- Relation between Kitaev chain and lattice quantum Hall model (Chern insulator, quantum anomalous Hall effect)
- Drive lower-dim. topological state to critical point \(\rightarrow\) massless state
- Couple massless states to get higher-dim. topological states
- QHE can be realized in translationally invariant lattice models without magnetic field
- Chiral edge states cannot exist by themselves in 1D: 1D model always has pairs of left \& right movers \(\rightarrow\) only way to realize only left movers is on the boundary of a higher-dimensional state

General principle: forbidden states in a lower dimension and nontrivial topological states in a higher dimension are related

Prototype lattice model for all sorts of topological states
Haldane model, Berry curvature, and Chern number

Recipe to obtain QHE: massless Dirac model + mass term that can change sign

Real 2D Dirac material: Graphene!

Haldane 1988 (before graphene was synthesized):
  give graphene a gap (mass term).

Graphene: 2D honeycomb lattice — triangular lattice with basis \( A \& B \)

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Lattice function in unit cell: vector \( (\psi_A, \psi_B)^T \) of amplitudes on \( A \& B \) sites.
Simple hopping model with hopping strength \( t \) along bonds between \( A \) and \( B \) sites:
\( (A \text{ and } B \text{ sublattices) } \)
\[ H_0(k) = \begin{pmatrix} 0 & h(k) \\ h^*(k) & 0 \end{pmatrix}, \quad k = (k_x, k_y) \]

\[ h(k) = t_n \sum_{i=1,2,3} \exp(i \mathbf{k} \cdot \mathbf{a}_i), \quad \mathbf{a}_i; \text{ see figure} \]

Using 2x2 Pauli matrices \( \sigma \) which act on sublattice space:

\[ H_0(k) = t_n \sum_i \left[ \sigma_x \cos(k \cdot a_i) - \sigma_y \sin(k \cdot a_i) \right] \]

Re and Im of \( \exp(i \mathbf{k} \cdot \mathbf{a}_i) \)

Energy dispersion:

\[ E(k) = \pm |h(k)| \rightarrow \text{graphene band structure} \]

with Dirac cones near \( \mathbf{K} = \left( \frac{2\pi}{3}, -\frac{2\pi}{3\sqrt{3}} \right) \)

\[ \mathbf{K}' = \left( \frac{2\pi}{3}, -\frac{2\pi}{3\sqrt{3}} \right) \]

Symmetries of graphene:

- Sublattice symmetry \( \sigma_z H_0(k) \sigma_z = -H_0(k) \)
  
  only approximate and consequence of only nearest-neighbor hopping; not important for Dirac cones

- Inversion symmetry: \( A \leftrightarrow D \) important for Dirac cones

- 3-fold rotational symmetry around center of unit cell, important for Dirac cones but not in the following.
* time-reversal symmetry: spinless $\rightarrow$ only complex conjugation in $k$ space
  $H_0(k) = H_0^*(-k)$
  $\rightarrow$ exchange the two Dirac cones

* product of sublattice (approximate in real graphene) and time-reversal symmetry
  $\Rightarrow$ particle-hole symmetry
  $\sigma_z H_0^*(-k) \sigma_z = -H_0(k)$

Make graphene topological:

$\rightarrow$ need bulk gapped
$\rightarrow$ break inversion and/or time-reversal symmetry

1) broken inversion symmetry; onsite potential $\pm M$ or $\pm iM$

$H_0(k) + M \sigma_z$

$\Rightarrow E(k) = \pm \sqrt{|h(k)|^2 + M^2}$ gapped

But: boring; for $M \gg t$, the electronic states are localized on either $A$ or $B$ sublattices; no trace of edge states

Why? $M \sigma_z$ preserves time-reversal symmetry

2) add imaginary second-neighbor hoppings:
• \( i t_2 \) hopping are purely imaginary and have chirality (handedness)
• they couple \( A-A \) and \( B-B \)

\[ H(k) = H_0(k) + M \sigma_z + 2t_2 \sum_i \sigma_i \sin (k \cdot \vec{b}_i) \]

\[ H_{t_2}^* (-k) = 2t_2 \sum_i \sigma_i \sin (-k \cdot \vec{b}_i) = \bigcirc H_{t_2} (k) \]

Note: there is also a \( \sin (k \cdot \vec{a}_i) \) term in \( H_0(k) \)
but it goes with \( \sigma_y \), which also changes sign under time reversal:
\[ \sigma_y^* = -\sigma_y \]

\( \Rightarrow \) \( H_0(k) \) does not break TRS!
band structure demo

\[ M \neq 0, \; t_2 = 0 : \text{boring gapped phase} \]
(there might be edge states depending on lattice termination, but these are not chiral, dispersionless, and boring; also they do not connect valence and conduction bands)

\[ t_2 \text{ passes through } \pm M/3\sqrt{3} : \text{gap closes and changes sign at one of the two Dirac points} \]
- at \( K' \) for \( t_2 = +M/3\sqrt{3} \)
- at \( K \) for \( t_2 = -M/3\sqrt{3} \)

\[ \rightarrow \text{chiral edge states appear!} \]
\[ \rightarrow \text{we have created a Chern insulator} \]

Test: What happens if we take a topological Haldane model and turn on a weak magnetic field?

A \[ \rightarrow \text{Landau levels, which change the number of edge states} \]

B \[ \rightarrow \text{if mag. field is weak, nothing changes if it does not close the gap} \]

C \[ \rightarrow \text{bulk gap closes and edge states disappear} \]

D \[ \rightarrow \text{gap does not close but edge states may change propagation direction depending on sign of field} \]
Pumping and Berry phase:

RHE with Laughlin argument: adiabatically piece flux $\Phi$ through RHE cylinder so that Hamiltonian returns to itself:

$$H(k) \rightarrow H\left(k + e\vec{A}\right) \quad \text{with} \quad \vec{A} = \frac{\partial \Phi}{\partial L}$$

Flux pumping $\rightarrow$ change in momentum

Flux change by integer number of flux quanta:

momentum $k$ changes by reciprocal lattice vector

$\rightarrow$ Bloch Hamiltonian returns to itself

Simplicity: Use a 2D square lattice zone

$$\left( k_x, k_y \right) \in \left[ 0, 2\pi \right] \quad \text{all arguments apply equally to the hexagonal graphene zone}$$

Adiabatic time evolution of an eigenstate $|\psi(k)\rangle$ of

$H(k)$ with energy $E(k)$ as $k$ is changed slowly

$|\psi(k)\rangle$ should remain non-degenerate (true for Haldane model)

$\rightarrow$ can follow adiabatically

Choose path $k(t)$ with $k(0) = k(T)$ — periodic

\[ \text{Diagram with a square lattice and a path through it.} \]
What is the final quantum state at \( T \)?

\[
\left| \psi(k_x, k_y + 2\pi) \right> \overset{?}{=} \left| \psi(k_x, k_y) \right> e^{-i \int_0^T E(k_x(t)) \, dt}
\]

No! Berry: Additional phase \( \gamma(C) = \oint_C \vec{A}(\vec{r}) \cdot d\vec{r} \)

\[
\vec{A}(\vec{r}) = \int \left( \psi(\vec{r}) \right| \nabla \psi(\vec{r}) \right> \\
\text{"Berry connection"}
\]

\[
\text{"Berry phase"}
\]

In our example \( \gamma \) will depend on \( k_x \), which depends on the path \( C \) but not on \( k_x(t) \), the actual “time evolution”. Berry phase = geometric phase

Later: \( \gamma \) is related to the topological character of the Hamiltonian and its ground state wavefunction.

**Flux pumping and Chern number:**

\( \gamma(k_x) \) info about change pumped during adiabatic cycle.

we know: pumped charge is invariant as long as energy gap is preserved.

\[
\text{we can change } E(k_x, k_y) \text{ if we do not close the gap}
\]

choice: flat \( E \) along \( k_x \) for fixed \( k_y \)

\[
\text{localized state in single unit cell in } x \text{ direction since all wave functions have same energy independent of } k_x
\]

\[
\left| \psi(n, t=0) \right> = \int dk_x e^{ik_x n} \left| \psi(k_x, k_y=0) \right>
\]
\[ |\psi(x, t=T)\rangle = \int_{-\frac{\pi}{2\theta}}^{\frac{\pi}{2\theta}} dl_x e^{i k_x l_x} e^{i \frac{\hbar}{\theta} (l_x - i \theta(l_x))} |\psi(x, l_y=0)\rangle \]

\[ \Theta(l_x) = \int_{0}^{T} dt \gamma(E(\theta(l_x) + 2\pi l_x)) \quad \text{phase} \]

While \( \Theta(l_x) \) truly periodic in \( l_x \) because \( E(l_x) = E(l_x + 2\pi) \), the only restriction on \( \gamma(l_x) \) is to be periodic modulo \( 2\pi \):
\[ \gamma(l_x + 2\pi) = \gamma(l_x) + 2\pi W, \quad W \in \mathbb{Z} \quad \text{integer} \]

Deform dispersion along \( l_y \) to make \( \gamma(l_x) - \Theta(l_x) \) a large as possible:
\[ \gamma(l_x) - \Theta(l_x) = W l_x \]

\[ |\psi(x, t=T)\rangle = \int dl_x e^{i k_x (n+W)} |\psi(x, l_y=0)\rangle \]

\[ \Rightarrow \] wave function shifted over by \( W \) unit cells.

\[ \Rightarrow \] pump \( W \) units of charge if Berry phase satisfies:
\[ \gamma(l_x + 2\pi) - \gamma(l_x) = 2\pi W \]

\( W \) is Chern number = topological invariant characterizing the band structure of 2D Quantum Hall Systems.

In fact: \( W = \text{bulk topological invariant} \)
Of all insulators with broken time-reversal symmetry, \( W = 0 \): trivial insulator no chiral edge states
\( W = n \): \( n \) chiral edge states, Chern insulator
Chern number from Berry curvature

Berry connection $\vec{A}(\vec{r}) \leftrightarrow$ vector potential $\vec{A}(\vec{r})$?

Similarities: • $\vec{A}(\vec{r})$ is gauge - dependent

$|\psi(\vec{r})\rangle \rightarrow e^{i\vec{A}(\vec{r})} |\psi(\vec{r})\rangle$

$\Rightarrow \vec{A}(\vec{r}) \rightarrow \vec{A}(\vec{r}) + \vec{\nabla}_r \lambda$

• Berry phase is gauge - independent for closed paths

Idea: In electromagnetism, $\vec{D} \times \vec{A}(\vec{r})$ is the magnetic field, which is measurable.

Here: $\vec{D}_c \times \vec{A}(\vec{r}) \equiv \vec{D}_c(\vec{r})$ Berry curvature

$= i\left[ \frac{\partial \psi}{\partial x} \left| \frac{\partial \psi}{\partial y} \right) - \left| \frac{\partial \psi}{\partial y} \frac{\partial \psi}{\partial x} \right] \right]$ (2-component \( \vec{B}_c \) for 2D)

$\Rightarrow \text{gauge - independent}$

Stokes theorem: Brillouin zone = 2-torus (\( \mathbb{B}_2 \))

$2\pi \lambda \equiv \lambda(2\pi) - \lambda(0)$

$= \iint_{\mathbb{B}_2} \vec{D}_c(\vec{r}) \cdot d\vec{S}$

$\equiv \text{Chern number defined in terms of } |\psi(\vec{r})\rangle \text{ only,}$

closed - form expression in terms of derivatives of $|\psi(\vec{r})\rangle$

via Berry curvature

$\lambda \neq 0$: "flux" coming out of a closed surface
\[ \rightarrow \text{ "magnetic monopole" (Dirac monopole)} \]

\[ \rightarrow \text{what are the sources of Berry curvature?} \]

\textbf{Gap closings are sources of Berry curvature}

\[ t_2/M \]

- Brillouin zone
- topological phase, \( W = 1 \)
- gap closing at \( K' \)
- negative Berry curvature
- trivial phase, \( W = 0 \)
- gap closing at \( K \)
- topological phase, \( W = -1 \)

\textit{massless cones are sources of Berry Curvature}
From Quantum Spin Hall effect to topological insulators

Adding symmetry to a topological insulator

Approaches to discover novel topological states:

(i) 

-low-D to higher-D to create edge states
(cf. Kitaev, Chern insulator)

(ii) 

-topology (K-theory): calculate topological classification from dimensionality and symmetry

-powerful but hard

(iii) 

-take one topological model and enforce additional symmetry

→ this will be used now

Example: 

-quantum dot with \( H_0 \)

- \( Q = \# \) filled levels

-enforce particle-hole symmetry

\[
H_{Q,d6} = \begin{pmatrix} H_0 & -H_0^* \\ H_0^* & -H_0 \end{pmatrix}
\]

→ topologically trivial w.r.t. \( Q = \text{const} \) here

but there are level crossings

→ are crossings protected when SC \( \Delta \) is added

to couple the blocks ?

→ answer (see (i)): yes! Pfaffian invariant
Now: add time-reversal symmetry to Chern insulator

CI: chiral edge states that are flipped under time-reversal $\mathcal{T}$

add tras: $H = \begin{pmatrix} H_0 & 0 \\ 0 & JH_0J^{-1} \end{pmatrix}$ (time-reversal simply flips the two blocks)

$H_0$: CI with $N$ edge states

$H$: $N$ pairs of counter-propagating edge states that transform into each other under time-reversal

$N=1$ sketched:

\[ \begin{array}{c}
\begin{array}{c}
\text{+}
\end{array}
\end{array} \]

next: are these edge states robust when the blocks are coupled?
A perfectly transmitted channel and Kramers degeneracy:

Scattering states:

Incoming \( |n_1, L\rangle \quad n=1, \ldots, N \)
\( |n_1, R\rangle \)

Outgoing \( T |n_1, L\rangle \) \( \quad \) time-reversed partners of incoming states
\( T |n_1, R\rangle \)

Scattering states:
\( |\Psi_{1, L}\rangle = \sum_{n=1}^{N} (\alpha_{n_1, L} |n_1, L\rangle + \beta_{n_1, L} T |n_1, L\rangle) \)
\( |\Psi_{1, R}\rangle = \sum_{n=1}^{N} (\alpha_{n_1, R} |n_1, R\rangle + \beta_{n_1, R} T |n_1, R\rangle) \)

Define vectors: \( \alpha_L \equiv (\alpha_{n_1, L}, \ldots, \alpha_{N,L}) \) etc.

Relation between incoming and outgoing modes: scattering matrix \( S \) of disordered region
\[
\begin{pmatrix}
\beta_L \\
\beta_R
\end{pmatrix} = S
\begin{pmatrix}
\alpha_L \\
\alpha_R
\end{pmatrix}, \quad S \text{ is } 2N \times 2N \text{ matrix}
\]
\( S = S^\dagger \) unitary (no losses)

Split \( S \) into \( N \times N \) reflection and transmission blocks:
\[
S = \begin{pmatrix}
r & t \\
t' & r'
\end{pmatrix}
\]
Can there be no transmission at all?
This would mean $t = t' = 0$ due to backscattering via disorder, gapped at edge states,...

If yes: all modes perfectly reflected back:

\[ r^+ r = r^+ r' = 1 \quad \text{(unitary blocks)} \]

\[ \rightarrow \text{ need to understand constraints imposed by time-reversal symmetry.} \]

**Scattering matrices with time-reversal symmetry:**

\[ \begin{pmatrix} \langle \mathcal{I}^+ | \mathcal{I} \rangle \\ = \langle \mathcal{I}^+ | \mathcal{I}^* \rangle \end{pmatrix} \]

Time reversal: anti-unitary operator $\mathcal{J}$

which commutes with $H$

**two flavors:** $\mathcal{J}^2 = \pm 1$

+ case: systems with no or integer spin

$\mathcal{J} = K$ complex conjugation

- case: systems with half-integer spin

simplest case $s = \frac{1}{2}$: $\mathcal{J} = i \sigma_y K$

Apply to scattering states:

\[ \mathcal{J} | \mathcal{I} \rangle \rangle = \sum_n \alpha_{n,\mathcal{I}}^* \mathcal{J} | n, \mathcal{I} \rangle + \beta_{n,\mathcal{I}}^* \mathcal{J}^2 | n, \mathcal{I} \rangle \]

\[ \mathcal{J} | \mathcal{I} \rangle \rangle = \sum_n \alpha_{n,\mathcal{I}}^* \mathcal{J} | n, \mathcal{I} \rangle + \beta_{n,\mathcal{I}}^* \mathcal{J}^2 | n, \mathcal{I} \rangle \]

$\mathcal{J}$ does not change the energy of states

\[ \rightarrow \text{ same scattering matrix as for } | \mathcal{I} \rangle \rangle, | \mathcal{I} \rangle \rangle \]

but roles of $\alpha, \beta$ are now reversed

\[ S \mathcal{J}^2 \begin{pmatrix} \alpha_{\mathcal{I}}^* \\ \beta_{\mathcal{I}}^* \end{pmatrix} = \begin{pmatrix} \alpha_{\mathcal{I}}^* \\ \beta_{\mathcal{I}}^* \end{pmatrix} \]
Multiply both sides by $J^2 S^T$ and take c.c.:

\[
\begin{pmatrix}
\alpha_n \\
\beta_n
\end{pmatrix} = J^2 S^T \begin{pmatrix}
\alpha_n \\
\beta_n
\end{pmatrix}
\]

\[
\Rightarrow \quad S = J^2 S^T
\]

If $J^2 = +1$ : $S = S^T$ symmetric

If $J^2 = -1$ : $S = -S^T$ anti-symmetric

Now try to set $t = t' = 0$:

For $S = S^T$, we cannot tell

But for $S = -S^T$ and $t = t' = 0$, we have

$r + r = 1$ unitary

$r = -r^T$ anti-symmetric

If $N$ odd: impossible — any odd-dimensional anti-symmetric matrix must have a single zero eigenvalue, while unitary matrices only have eigenvalues with unit norm!

$\Rightarrow t = 0$ is impossible!

$\Rightarrow$ zero eigenvalue of $r$ means that there is always a single mode that is transmitted with unit probability !!!
If $J^2 = -1$ and number of edge states going in one direction is odd, they cannot be gapped out. If there is an even number of edge states, they can be gapped out.

Since these are the only two options, the topological invariant of a Chern insulator is reduced to a $\pm 1$ invariant ("$\mathbb{Z}_2$ invariant") in the presence of time-reversal symmetry. The topologically protected counter-propagating edge states are called helical edge states.

Helical edge states are Kramers pairs

Why is $J^2 = -1$ special?

Kramers Theorem:

For every energy eigenstate of a time-reversal symmetric system with half-integer spin, there is at least one more eigenstate with the same energy.

Proof: $\text{Trs} \Rightarrow [\mathcal{H}, T] = 0$. If $|n\rangle$ eigenstate with $\mathcal{H} |n\rangle = E_n |n\rangle$, then $J |n\rangle$ eigenstate and $H T |n\rangle = J H |n\rangle = J E_n |n\rangle = E_n J |n\rangle$. Crucial ingredient: $J |n\rangle$ and $|n\rangle$ are different.
States for half-integer spin! $\hat{J}$ reverses all angular momenta, and the magnetic quantum number (e.g., $S_z$ for $S=\frac{1}{2}$) is never zero! Hence, $S_z = \frac{1}{2} \rightarrow S_z = -\frac{1}{2}$ under $\hat{J}$. Q.E.D.

$|\psi\rangle$ and $J|\psi\rangle$ are orthogonal: $\langle \psi | J | \psi \rangle = 0$

Kramers pair

$\Rightarrow$ it is impossible to introduce any backscattering between them without breaking trs!

Example: $N=1$, $S=\frac{1}{2} \Rightarrow S_z = \pm \frac{1}{2} \uparrow$ and $\downarrow$

$\Rightarrow$ reflection requires spin-flip scattering

$\Rightarrow$ forbidden by trs

$\Rightarrow$ perfect transmission
The quantum spin Hall effect

2D topological insulator with time-reversal symmetry

= "\mathbb{Z}_2" topological insulator — only indicates that topological invariant can only take two values ±1

better name: "quantum spin Hall insulator"

- electrons injected from left (1)
- same # of channels connecting 1-3 and 1-4
  => no net Hall current
  => \( \sigma_{\text{Hall}} = 0 \) as expected for TRS system
- but: \( \rightarrow \) and \( \leftarrow \) have opposite spin
  => there may be a net spin current
- if \( S_z \) preserved: \( \rightarrow \) have \( \uparrow \)-spin
  \( \Rightarrow \ up \) go from 1 to 4
  \( \downarrow \) go from 1 to 3
  \( \Rightarrow \) quantized spin current between 3 and 4
Quantum spin Hall effect

Not a general property of $\mathbb{Z}_2$ topological insulator — only when spin conservation law is present.

Model for quantum spin Hall (QSH) insulator

Bernevig - Hughes - Thiang (BHT) model:

- 2 copies of Chern insulator on square lattice

\[
H_{BHT}(\vec{k}) = \begin{pmatrix}
    h(\vec{k}) & 0 \\
    0 & h^*(\vec{k})
\end{pmatrix}
\]

\[
h(\vec{k}) = \epsilon(\vec{k}) \mathbb{1} + \vec{d}(\vec{k}) \cdot \vec{\sigma}
\]

\[
\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z), \text{ Pauli matrices act on electron/hole degrees of freedom of the Chern insulator.}
\]

\[
\epsilon(\vec{k}) = C - D \left( k_x^2 + k_y^2 \right)
\]

\[
\vec{d}(\vec{k}) = (A k_x, -A k_y, M(\vec{k}))
\]

\[
M(\vec{k}) = M - B \left( k_x^2 + k_y^2 \right)
\]

$A k_x \sigma_x - A k_y \sigma_y$: linear coupling between hole and electron

$M(\vec{k})$: momentum-dependent mass
M change sign: gap closing at $b^2 = 0$.

trivial $\rightarrow$ topological insulator.

BHZ model applies to semiconductor sandwich materials involving strong spin-orbit coupling.
Before we have derived the Chern number \( W \) as a topological invariant related to the Berry curvature of filled energy bands of periodic systems. We have also argued that \( W \) counts the number of chiral edge modes in a Chern insulator.

Here: Explicit calculation of the Hall conductivity

\[
\Sigma_{xy} = W \frac{e^2}{h}
\]

quantized in units of the conductance quantum \( \frac{e^2}{h} \) from the Kubo formula (linear response).

Linear response in a nutshell:

\[
H = H_0 + \Delta H
\]

\[
\begin{array}{c}
\uparrow \\
\text{system}
\end{array} \quad \begin{array}{c}
\uparrow \\
\text{perturbation}
\end{array}
\]

Energy eigenstates of \( H_0 \): \( H_0 |n> = E_n |n> \) (formal \( h \) general: unknown)

Electric field: \( \vec{E} = -\partial_t \vec{A} \)

\[
\rightarrow \Delta H = -\vec{j} \cdot \vec{A}
\]

Choose: \( \vec{A}^0(t) = \frac{\vec{E}^2}{i\omega} e^{-i\omega t} \) (take DC limit \( \omega \to 0 \))

Note: \( \vec{A}^0 \) terms only matter for longitudinal transport, reflected here.
**Goal:**

\[
\langle \mathcal{J}(\omega) \rangle = \frac{\sigma^2(\omega)}{\mathcal{R}} \bar{E}(\omega)
\]

Only depend linear field on \(H_0\)

\(\sigma^2(\omega)\): tensor of optical conductivity

More specific: \(\sigma_{xy}(\omega)\) Hall conductivity

**Interaction picture:** \(\mathcal{O}(t) = V^{-1} \mathcal{O} V, \quad V = e^{-iH_0 t / t}\)

\[
|\psi(t)\rangle = U(t, t_0) |\psi(t_0)\rangle
\]

\[
U(t, t_0) = T \exp \left( -\frac{i}{\hbar} \int_{t_0}^{t} \Delta H(t') \, dt' \right)
\]

Hermitian ordering: \(i \hbar \dot{U} = \Delta H U\)

\[
U(t) = U(t, t_0 \rightarrow -\infty), \text{ prepare system in } |0\rangle \text{ at } t_0 \rightarrow -\infty: \text{(usually: ground state)}
\]

\[
\langle \mathcal{J}(t) \rangle = \langle 0| U^{-1}(t) \mathcal{J}(t) U(t) |0\rangle
\]

\[
\approx \langle 0 | \left( \mathcal{J}(t) + \frac{i}{\hbar} \int_{-\infty}^{t} dt' \left[ \Delta H(t'), \mathcal{J}(t') \right] \right) |0\rangle
\]

Expansion of \(U(t)\) keeping only leading terms.

**Assume:** \(\langle 0 | \bar{\mathcal{J}}(t) |0\rangle = 0\) no current in ground state

\[
\langle \mathcal{J}_i(t) \rangle = \frac{1}{\hbar \omega} \int_{-\infty}^{t} dt' \langle 0 | \left[ \bar{\mathcal{J}}_i(t'), \mathcal{J}_i(t) \right] |0\rangle E_j e^{-i\omega t'} = \ldots
\]

Time-transf. inv.: depends only on \(t'' \equiv t - t'\)

\[
\ldots = \frac{1}{\hbar \omega} \left( \int_{0}^{\infty} dt'' e^{i\omega t''} \langle 0 | \left[ \bar{\mathcal{J}}_i(0), \mathcal{J}_i(t'') \right] |0\rangle E_j e^{-i\omega t''} \right)
\]

Actually: \(\omega \rightarrow \omega + \imath 0^+\) for convergence
Hall conductivity:

\[ \sigma_{xy}(\omega) = \frac{i}{\hbar} \sum_{n \neq 0} \left( \frac{\langle 0| J_y | n \rangle \langle n| J_x | 0 \rangle - \langle 0| J_x | n \rangle \langle n| J_y | 0 \rangle}{\hbar \omega + E_n - E_0} \right) \]

**Kubo formula**

Inserting \( H_0 |n\rangle = E_n |n\rangle \) eigenstates:

\[ \sigma_{xy}(\omega) = -\frac{1}{\hbar \omega} \sum_{n \neq 0} \left( \frac{\langle 0| J_y | n \rangle \langle n| J_x | 0 \rangle - \langle 0| J_x | n \rangle \langle n| J_y | 0 \rangle}{\hbar \omega + E_n - E_0} \right) \]

DC \( \omega \to 0 \) limit:

\[ \frac{1}{\hbar \omega + E_n - E_0} \approx \frac{1}{E_n - E_0} - \frac{\hbar \omega}{(E_n - E_0)^2} + \cdots \]

Only important for longitudinal conductivity, vanishes for Hall

\[ \sigma_{xy} = i\hbar \sum_{n \neq 0} \frac{\langle 0| J_y | n \rangle \langle n| J_x | 0 \rangle - \langle 0| J_x | n \rangle \langle n| J_y | 0 \rangle}{(E_n - E_0)^2} \]

**Kubo formula for Hall conductivity**

**Role of topology**

Consider particles on lattice. Brillouin zone:

\[ -\frac{\pi}{a} < k_x \leq \frac{\pi}{a}, \quad -\frac{\pi}{b} < k_y \leq \frac{\pi}{b} \]

Since reciprocal lattice periodic:

\[ \pi \leq k_x, \quad \pi \leq k_y \]

\[ \rightarrow \pi^2 \]
wave functions (Bloch): $\psi_{k}^{x}(x) = e^{ik_{x}x}$ periodic in $k_{x}$

- assume that single-particle spectrum decomposes into separate bands $E_{n}(k_{x})$
- noninteracting electrons: fill single-particle states employing only Pauli principle
- assume excitation gap between bands and Fermi energy $E_{F}$ in one of these gaps

$\Rightarrow$ band insulator $E_{n}(k_{x})$

$\Rightarrow$ assign integer-valued topological invariant $\nu \in \mathbb{Z}$ to each band
$U(1)$ Berry connection defined over $T^2$:

$$A_j(x^i) = -i \langle u_{k'} \left| \frac{\partial}{\partial x^j} \right| u_k \rangle$$

\(\rightarrow\) field strength (Berry curvature):

$$F_{xy} = \frac{\partial A_x}{\partial y} - \frac{\partial A_y}{\partial x} = -i \left( \langle u_{k'} \left| \frac{\partial u_{k'}}{\partial y} \right| u_k \rangle + \langle u_{k'} \left| \frac{\partial u_{k'}}{\partial x} \right| u_k \rangle \right)$$

\(\rightarrow\) 1st Chern number:

$$W = \frac{1}{2\pi} \int_{T^2} d^2 k \ F_{xy} \quad \text{(TKNN invariant)}$$

Thouless, Kohno, Nightingale, and Nielsen, PRL 49, 405 (1982)

Kubo formula for Hall conductivity using tensor product of single-particle wavefunctions:

$$\Sigma_{xy} = i \tau \sum_{E_\alpha < E_\beta < E_\gamma} \int \frac{d^2 k}{(2\pi)^2} \frac{\langle u_\alpha^x \mid J_y \mid u_\beta^y \rangle \langle u_\beta^y \mid J_x \mid u_\gamma^x \rangle - \langle x \leftrightarrow y \rangle}{(E_\gamma(k) - E_\alpha(k))^2}$$

$\alpha$: filled bands

$\beta$: empty bands

(actually: separate momentum integrals... but will not matter here)

Current:

$$\langle \psi_k \mid H \mid \psi_k \rangle = E_k \langle \psi_k \rangle$$

\(\Rightarrow\) \(e^{-i k x} H e^{i k x} \mid u_k \rangle = E_k \mid u_k \rangle$$

\(\Rightarrow\) \(\tilde{H}(k) \mid u_k \rangle = E_k \mid u_k \rangle\), \(\tilde{H}(k) = e^{-i k x} H e^{i k x}\)

Current:

$$\vec{J} = \frac{e}{\hbar} \frac{\partial \tilde{H}}{\partial k} \in \text{group velocity of wavepackets}$$

Check: continuum $H = (p + eA)^2/2m$
\[ \hat{H} = \frac{(p + t x + e A)^2}{2 m} \quad \Rightarrow \quad \hat{J} = e \hat{x} \times \hat{A} \]

\[ \sigma_{xy} = \frac{ie^2}{h} \sum_{\text{filled}} \int \frac{d^2 k}{(2\pi)^2} \left< \partial_y u^\alpha \hat{H} \partial_x u^\alpha \right> - \left( \alpha \leftrightarrow \beta \right) \]

\[ \partial_x, \partial_y \text{ means } \frac{\partial}{\partial k_x}, \frac{\partial}{\partial k_y} \]

\[ \left< u^\alpha_k \hat{H} u^\beta_k \right> = \left< u^\alpha_k \partial_i \hat{H} u^\beta_k \right> - \left< u^\alpha_k \partial_i \hat{H} u^\beta_k \right> \]

\[ = - (E^\beta_k - E^\alpha_k) \left< u^\alpha_k \partial_i u^\beta_k \right> \]

\[ \Rightarrow \sigma_{xy} = \frac{ie^2}{h} \sum_{\text{filled}} \int \frac{d^2 k}{(2\pi)^2} \left< \partial_y u^\alpha_k \partial_x u^\alpha_k \right> - \left( \alpha \leftrightarrow \beta \right) \]

\[ \text{Sum over empty bands: } \sum_{\beta} \left| u^\beta_k \right|^2 \left< u^\beta_k \right| u^\alpha_k \rangle = 1 - \frac{\sum_{\alpha} \left| u^\alpha_k \right|^2 \left< u^\alpha_k \right| u^\alpha_k \rangle}{\text{vanishes by symmetry}} \]

\[ \Rightarrow \sigma_{xy} = \frac{ie^2}{h} \sum_{\text{filled}} \int \frac{d^2 k}{(2\pi)^2} \left< \partial_y u^\alpha_k \partial_x u^\alpha_k \right> - \left< \partial_x u^\alpha_k \partial_y u^\alpha_k \right> \]

\[ \sigma_{xy} = \frac{e^2}{h} \sum_{\alpha} W^\alpha \]

- **Kubo = Chem → Japanese Physicist = Chinese Mathematician**
- **TKNN Formula: Hall Conductivity = Sum of root**
- **Chern Number of filled bands = Number of chiral edge modes of Chern Insulator / QH Insulator**
Chern insulator revisited

\[ \hat{H}(\mathbf{k}) = \hat{h}(\mathbf{k}) \cdot \mathbf{\sigma}^2 + \varepsilon(\mathbf{k}^2) 1 \]

\( \mathbf{k} \in \mathbb{R}^2 \), \( \mathbf{\sigma} = (\sigma_x, \sigma_y, \sigma_z) \) Pauli matrices

energies : \( \varepsilon(\mathbf{k}^2) = \varepsilon(\mathbf{k}) \pm |\hat{h}(\mathbf{k}^2)| \)

For example : \( \hat{H}(\mathbf{k}) = \sin h_x \sigma_x + \sin h_y \sigma_y + (m + \cos k_x + \cos k_y) \sigma_y \)

Xiao-Liang Qi, Yong-Shi Wu, Shou-Cheng Zhang. cond-mat/0503058

Pseudospin vector : \( \mathbf{\vec{n}}(\mathbf{k}) = \frac{\mathbf{\hat{n}}(\mathbf{k})}{|\hat{h}(\mathbf{k}^2)|} \)

\( \mathbf{\vec{n}} \in S^2 \equiv 2\text{-sphere} \equiv \text{Bloch sphere} \):

Mapping from \( \mathbb{R}^2 \to S^2 \)

| \mathbb{R}^2 | \rightarrow | S^2 |

Final exercise: find band closings for \( \hat{H}(\mathbf{k}) \)

(values of \( m \))

compute the Chern number in the different phases

What does this have to do with the winding of \( \mathbf{\vec{n}} \) around the Bloch sphere?