Classical limit

to perform the classical limit, we restore the units $\hbar$ (which was just set $\hbar = 1$) and then take "$\hbar \to 0$"

1) free particle, no dissipation

$$S = \frac{1}{\hbar} \int_{-\infty}^{+\infty} dt \left[-X_q \ddot{X}_{cl} - V(X_{cl} + X_q) + V(X_{cl} - X_q) \right]$$

Expectation value of any observable $O[X]$:

$$\langle O[X](t) \rangle = \frac{1}{Z_0} \int \mathcal{D}(X)\, e^{iS}\, O[X(t_+)]$$

$$= \text{——} \cdot \text{——} \; O[X(t_-)]$$

$$= \frac{1}{Z_0} \int \mathcal{D}(X_{cl}) \mathcal{D}(X_q)\, e^{i S[X_{cl}, X_q]}\, O(X_{cl}(t))$$

variable transformation (rescaling) $X_q \to X_q \hbar$

transformation of measure ($\hbar^{\#\text{ time steps}}$) absorbed in normalization constant $Z_0' = \int \mathcal{D}(X_{cl} X_q)\, e^{i S[X_{cl}, X_q \hbar]}$

$$\langle O \rangle = \frac{1}{Z_0'} \int \mathcal{D}(X_{cl}) \mathcal{D}(X_q)\, e^{i S[X_{cl}, X_q \hbar]}\, O(X_{cl}(t))$$

$$S[X_{cl}, \hbar X_q] = \frac{1}{\hbar}\int dt\left[-\hbar 2 X_q \ddot{X}_q - V(X_{cl}+\hbar X_q) + V(X_{cl}-\hbar X_q)\right]$$

$$= \int dt \left(-2 X_q \ddot{X}_{cl} - 2V'(X_{cl}) X_q + O(\hbar)\right)$$

one can now set $\hbar = 0$, and perform the X_q - integral

$$\int \mathcal{D}(X_q)\, e^{-i\int dt 2 X_q (\ddot{X}_{cl} + V'(X_{cl}))} \overset{\text{discret } X_{q,m},\ X_{cl,m},\ m=1\dots N}{=}$$

$$= \int \prod_m dX_{qm}\; e^{-i 2\Delta t_m X_{qm}(\ddot{X}_{cl,m} + V'(X_{cl,m}))}$$

$$= \prod_m \left(\frac{2\pi}{2\Delta t_m}\right) \delta(\ddot{X}_{cl,m} + V'(X_{cl,m}))$$

$\hookrightarrow$ absorbed in normalization constant

$$\sim \delta(\ddot{X}_{cl}(t) + V'(X_{cl})) \qquad (\hat{=}\ \ddot{X}_{cl} + V'(X_{cl})$$

enforced at any time

$$\langle O \rangle = \frac{1}{Z_0'} \int \mathcal{D}(X_{cl})\, \delta(\ddot{X}_{cl}(t) + V'(X_{cl}))\, O(X_{cl}(t))$$

$\hat{=}$ $O(t)$ computed along classical trajectory, i.e. where X satisfies Newton $\ddot{X} = -V'(X)$.

(... and that's why X_{cl} is called "classical")

2) dissipative environment

$\coth\left(\frac{\beta\omega}{2}\right)$ in $\Sigma^R_{bath} \rightarrow \coth\left(\frac{\beta\hbar\omega}{2}\right)$

$\hat{=}$ $T \rightarrow T/\hbar$ in final expression for S_{eff}.

with rescaling $X_q \rightarrow X_q \hbar$:

$$S_{eff} = \frac{1}{\hbar} i 2\gamma \int dt \left[2T/\hbar (\hbar X_q)^2 + \frac{\pi}{2}\left(\frac{T}{\hbar}\right)^2 \int dt' \frac{\hbar^2 (X_q(t) - X_q(t'))^2}{\sinh^2\left(\pi \frac{T(t-t')}{\hbar}\right)} \right]$$

to take $\hbar \rightarrow 0$ limit, use:

$$\frac{\pi/2\, T/\hbar}{\sinh^2\left(\pi \frac{T}{\hbar}(t-t')\right)} \rightarrow \delta(t-t') \quad \rightsquigarrow \text{last term drops out.}$$

$\Rightarrow$ (including all terms, see S_{eff} on page ..)

$$S_{eff} = \int_{-\infty}^{+\infty} dt \left\{ (-2X_q)\left[\ddot{X}_{cl} + \gamma \dot{X}_{cl} + V'(X_{cl}) \right] + 4i\gamma T X_q^2 \right\}$$

the quadratic term $\sim X_q^2$ can be transformed to a linear term by introducing an auxiliary field $\xi(t)$. This is a standard tool in field theory (Hubbard-Stratonovich transformation)

$$e^{i\int dt\ 4i\gamma T\, X_q^2} \simeq e^{-\sum_k \Delta t_k 4T\gamma\, X_{qk}^2}$$

Gaussian identity

$$\int d\xi\ e^{-\frac{\Delta t_k}{4T\gamma}\xi^2 + i\Delta t_k \xi x 2}$$

$$= \int d\xi\ e^{-\frac{\Delta t}{4T\gamma}\left(\xi^2 - 2\xi i 4T\gamma x + (i4T\gamma x)^2 - (i4T\gamma x)^2\right)}$$

$$= e^{-4T\gamma\Delta t x^2} \int d\xi\ e^{-\frac{\Delta t}{4T\gamma}(\xi - i4T\gamma 4)^2}$$

$$= e^{-4T\gamma\Delta t x^2} \int d\xi\ e^{-\frac{\Delta t}{4T\gamma}\xi^2}$$

$$\Rightarrow \boxed{e^{-\int dt\ 4\gamma T\, X_q^2} = \int \mathcal{D}(\xi(t))\, e^{-\int dt \left(\frac{\xi(t)^2}{4\gamma T} - 2i\xi(t) X_q(t)\right)}}$$

with the normalization $\int \mathcal{D}(\xi)\, e^{-\int dt \frac{\xi(t)^2}{4\gamma T}} = 1$

(measure for $\mathcal{D}(\xi(t))$)

$$\Rightarrow \langle O \rangle = \int \mathcal{D}(\xi(t))\, e^{-\frac{1}{\gamma T 4}\int dt\, \xi(t)^2} \int \mathcal{D}(X_{ce})\, \mathcal{O}(X_{ce}) \times \int \mathcal{D}(X_q)\, e^{-i2\int dt\, X_q\left(\ddot{X}_{ce} + \gamma \dot{X}_{ce} + V'(X_{ce}) - \xi(t)\right)}$$

$$= \int \mathcal{D}(\xi(t))\, e^{-\frac{1}{4\gamma T}\int dt\, \xi^2} \int \mathcal{D}(X_{ce})\, \mathcal{O}(X_{ce})\, \delta\left(\ddot{X}_{ce} + \gamma \dot{X}_{ce} + V'(X_{ce}) - \xi(t)\right)$$

$\Rightarrow \langle O \rangle$ obtained by average over all path $X_\alpha(t)$, where $X_\alpha(t)$ satisfies the Newton equation of motion with friction γ and a "random force" $\xi(t)$:

$$\ddot{X}_\alpha = -\gamma \dot{X}_\alpha - V'(X_\alpha) + \xi(t)$$ Langevin equation

where the force $\xi(t)$ is chosen with relative probability $e^{-\frac{1}{4\gamma T}\int dt\, \xi(t)^2}$

(normalized to $\int \mathcal{D}(\xi(t))\, e^{-\frac{1}{4\gamma T}\int dt\, \xi(t)^2} = 1$)

- Because of the gaussian distribution of $\xi(t)$, the distribution is determined by its first and second moment only:

$$\langle \xi(t) \rangle = \int \mathcal{D}(\xi)\, e^{-\frac{1}{4\gamma T}\int dt\, \xi(t)^2}\, \xi(t) = 0$$

$$\langle \xi(t)\, \xi(t') \rangle = \int \mathcal{D}(\xi)\, e^{-\frac{1}{4\gamma T}\int \xi(t')^2 dt'}\, \xi(t)\, \xi(t')$$

$$\overset{\text{check}}{=} \delta(t-t')\, 2\gamma T$$

(check in discrete version)

$$\langle \xi_k\, \xi_{k'} \rangle \equiv \frac{1}{\text{Norm}} \int \prod_\ell d\xi_\ell\, e^{-\frac{1}{4\gamma T}\sum_\ell \Delta t_\ell \xi_\ell^2}\, \xi_k\, \xi_{k'} = \frac{2\gamma T\, \delta_{kk'}}{\Delta t_k}.$$

classical stochastic differential equations

Langevin equation defines "stochastic process" in continuous time. In this section we highlight some general properties of such stochastic differential equations, and their link to a path integral formalism.

- Stochastic process

  consider sequence of timesteps $t_1 < \dots < t_N$, and a random state variable X_i at each time t_i (ie., in principle $(X_1, \dots X_N)$ is a multi-dimensional random variable)

  stochastic process: Probability $p(x_n, t_n; \dots; x_1 t_1)$ determined by distribution at earlier times through some probabalistic rule, which is given by specifying the conditional probabilities $p(x_n, t_n \mid x_{n-1} t_{n-1}, \dots, x_1 t_1)$

important special case: Markov process:

transition rule depends only on one earlier timestep: $p(x_n t_n | x_{n-1} t_{n-1}, \dots) \equiv p(x_n t_n | x_{n-1} t_{n-1})$

$$= W_{t_{n-1}}(x_n | x_{n-1})$$

~> This implies that the distribution function is entirely determined by the initial state $x_1 t_1$; and we have the following composition rule: (Kolmogorov – Chapman)

$$p(x,t | x_1 t_1) = \int dx' \, p(xt | x't') \, p(x't' | x_1 t_1)$$

Example: Gaussian (diffusion) process:

discrete space $X \in \{0, \pm 1, \pm 2, \dots\} = \{n\}$

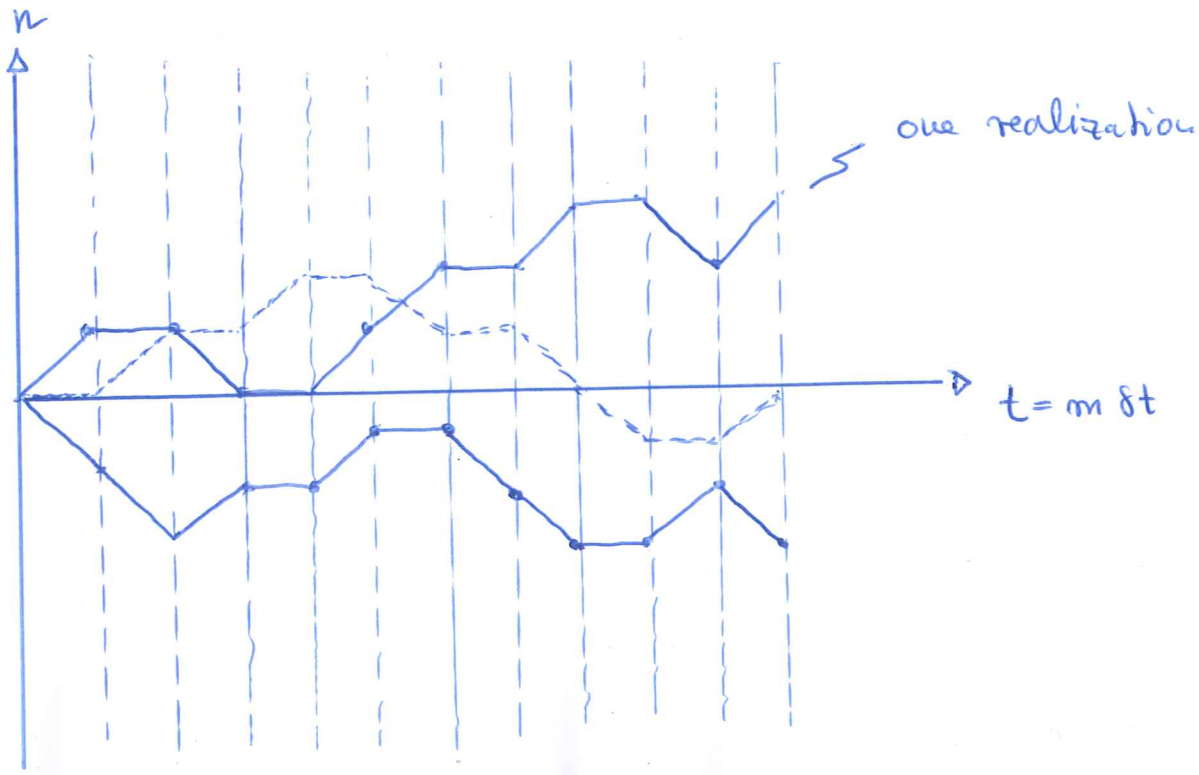
$$W(n,n') = (1-\delta t)\,\delta_{nn'} + q\,\delta t\,\delta_{n,n'+1} + (1-q)\,\delta t\,\delta_{n,n'-1}$$

$(0 < \delta t < 1, \quad 0 < q < 1)$

$q\,\delta t$: , $(q-1)\,\delta t$: propability to move left / right

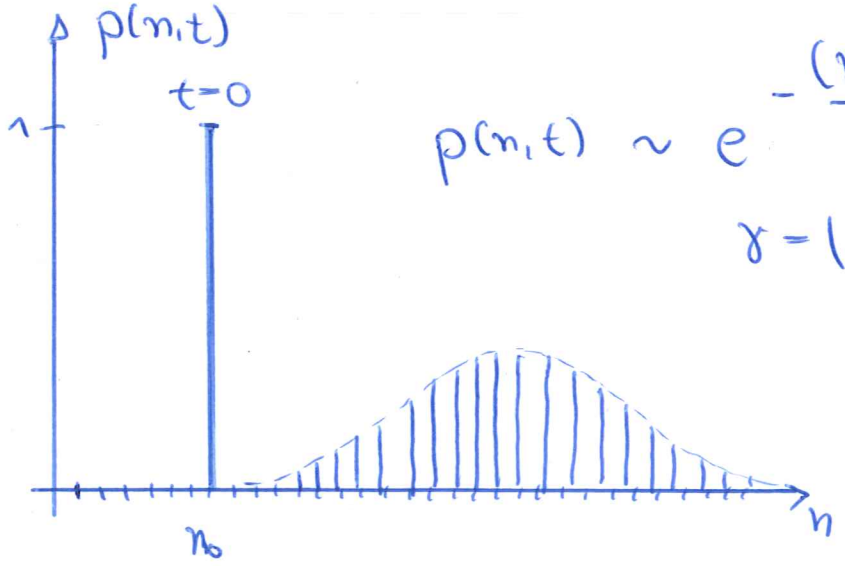
$\sum_n W(n|n') = 1$

$$\rightarrow \quad p(n, t+\delta t) - p(n,t) = \delta t \left[q \cdot p(n-1,t) + (q-1)\, p(n+1,t) - p(n,t) \right]$$



this process is easily discussed in

"continuum limit" $\delta t \to 0$ $t = m\delta t$ fix



$$p(n,t) \sim e^{-\frac{(n - n_0 - \gamma t)^2}{2t}}$$

$$\gamma = (1 - 2q)$$

proof: (exercise) : assume $p(n \pm 1, t)$ can be expanded in Taylor series around $p(n,t)$, solve differential equation for $p(x,t)$.

Stochastic differential equation (SDE)

Here we consider the generalized Langevin equation

$$\dot{x} = f(x) + g(x)\,\xi$$

$$\langle \xi(t)\,\xi(t')\rangle = A\,\delta(t-t') \qquad \langle \xi(t)\rangle = 0$$

} Gaussian random noise

(*)

Note:
- if $g(x)$ depends on x, the noise term is called multiplicative noise
- x, ξ, may be multi-dimensional. Then the equation above includes also high-order differential equations, e.g.

$$\ddot{x} = -V'(x) + \xi \quad \hat{=}$$

$$\begin{cases} \dot{x} = v \\ \dot{v} = -V'(x) + \xi \end{cases}$$

SDE $\hat{=}$ continuum limit of some stochastic process. Problem: $x(t)$ probably not differentiable for most realizations $\xi(t)$! (see Gaussian process above)

⇒ SDE continuum limit of specific discretization of the integral version of the differential equation

Itō - discretization of the Langevin equation:

$t \to i\delta t$, $i = 0, 1, 2, \ldots$, $\delta t \to 0$

$$X_i = X_{i-1} + \left[f(X_{i-1}) + g(X_{i-1})\, \xi_{i-1} \right] \delta t$$

ξ_i Gaussian random number,

$$\langle \xi_i \rangle = 0 \qquad \langle \xi_i \xi_j \rangle = A \delta_{ij} / \delta t$$

Other ways of writing this:

- Define Brownian motion Y_i as stochastic process generated by $Y_{i+1} = Y_i + \delta t\, \xi_i$, with ξ_i as above
  [$p(y, 0) = \delta(y)$ start from 0]
- For any other stochastic process X_i, define stochastic integral in Itō - discretization

$$\int_0^t X\, dY = \lim_{n \to \infty} \sum_{i=1}^{n} (Y_i - Y_{i-1})\, X_{i-1} \qquad (t = n \delta t)$$

Then the Langevin equation is written as

$$X(t) = X(0) + \underbrace{\int_0^t d\bar{t}\, f(X(t))}_{\text{usual integral}} + \underbrace{\int_0^t g(X)\, dY}_{\text{stochastic, Ito}}$$

differential notation:

$$\boxed{dX = f(X)\, dt + g(X)\, dY}$$

Examples: (see exercise)

Ornstein-Uhlenbeck process:

$$dX = -\gamma X dt + A\, dY \qquad \gamma, A \text{ const.}$$

geometrische Brownsche Bewegung → Black Scholes modell

$$dX = aX dt + bX dY$$

Note: If X_i and Y_i are dependent in the stochastic integral, then the result does depend on the precise discretization. Other discretizations, cite the "midpoint rule" (Stratonovich integral)

$$\int_0^t X dY \underset{\text{Stratonovich}}{=} \lim_{n\to\infty} \sum_{i=1}^{n} \frac{X_i + X_{i-1}}{2} (Y_i - Y_{i-1})$$

give a different result!

Therefore, also the Langevin equation with multiplicative noise is well defined only with a specific discretization.

Example: Ito vs Stratonovich discretization of the stochastic integral $\int_0^t y\,dy$.

Stratonovich:

$$\sum_{i=1}^{n} \frac{1}{2}(y_i + y_{i-1})(y_i - y_{i-1}) = \sum_{i=1}^{n} \frac{1}{2}(y_i^2 - y_{i-1}^2)$$

$$= \frac{1}{2}\left(y_n^2 - \underbrace{y_0^2}_{=0}\right) = \frac{1}{2} y_n^2$$

(mean square displacement of Brownian motion $\langle y_n^2 \rangle \sim t$)

Ito: $\sum_{i=1}^{n} y_i (y_i - y_{i-1}) =$

$$= \sum_{i=1}^{n} \frac{1}{2}\left(y_i^2 - y_{i-1}^2 - (y_i - y_{i-1})^2 \right)$$

$$= \frac{1}{2}\left(y_n^2 - y_0^2\right) - \frac{1}{2}\sum_{i=1}^{n} \underbrace{(y_i - y_{i-1})^2}_{\delta t \xi_{i-1}^2}$$

increments $\Delta y_i = y_i - y_{i-1}$ stat. independent with $\langle (\Delta y_i)^2 \rangle$

$= \delta t^2 \langle \xi_{i-1}^2 \rangle = A\,\delta t$

$$\left\langle \frac{1}{2}\sum_{i=1}^{n}(y_i - y_{i-1})^2 \right\rangle = \frac{An\delta t}{2} = \frac{At}{2}$$

using law of large numbers, one can for $n \to 0$ even replace $\frac{1}{2}\sum (y_i - y_{i-1})^2$ by average $= 0$

$$\int_{\text{Ito}} y\,dy = \int_{\text{Stratonovich}} y\,dy + At/2$$

Path-integral representation of the (Ito) Langevin equation

Consider discretized form of the SDE;
and let $x[\xi]$ be the path corresponding to a given realization ξ_i, i.e. the one which satisfies

$$x_i - x_{i-1} = f(x_{i-1}) + g(x_{i-1})\,\xi_{i-1}$$

The average of an observable can be written as

$$\langle O \rangle = \int \prod_{i=0}^{N} d\xi_i \; e^{-\sum_i \delta t\, \xi_i^2/2A}\; O_\xi[\xi]$$

$$O[\xi] = \int \left(\prod_{i=0}^{N} dx_i\right) O[\{x\}] \left(\prod_i \delta(x_i - x[\xi]_i)\right)$$

transformation of the N-dim δ-function:

$$\vec{R}(\vec{x}) : \mathbb{R}^n \to \mathbb{R}^n \qquad \vec{R}(\vec{x}_0) = 0$$

$$\delta^{(n)}(\vec{R}(\vec{x})) = \frac{1}{|\det(\partial R_i / \partial x_j)|}\, \delta^n(\vec{x} - \vec{x}_0)$$

transformation $R_i = x_i - x_{i-1} - \left[f(x_{i-1}) + g(x_{i-1}) \cdot \xi_{i-1}\right] \delta t$

$$R_0 = x_0 - \underbrace{x[\xi]_0}_{=0}$$

$\rightsquigarrow$ $R_i = 0 \;\forall i \Leftrightarrow x_i = x(\xi)_i$

$|\det(\partial R_i / \partial x_j)| = 1$

upper triangular matrix with 1 on diagonal $\Rightarrow$ Note: would be different for different discretization than Ito!

$$\rightsquigarrow O[\xi] = \int \left(\prod_i dx_i\right) O[x] \left(\prod_j \delta^{(n)}\left(x_j - x_{j-1} - \left[f(x_{j-1}) + g(x_{j-1})\xi_{j-1}\right]\delta t\right)\right)$$

$$= \int \left(\prod_i dx_i\right) O[x] \int \left(\prod_i \frac{dy_i}{2\pi}\right) e^{-i \sum_\ell y_\ell \left[x_\ell - x_{\ell-1} - \delta t \left[f(x_{\ell-1}) + g(x_{\ell-1})\xi_{\ell-1}\right]\right]}$$

average over disorder:

$$\frac{1}{N} \int d\xi\, e^{-\frac{1}{2A}\delta t \xi^2} e^{+i y_\ell \delta t g(x_{\ell-1})\xi}$$

$$= e^{-\delta t \frac{A}{2} g(x_{\ell-1})^2 y_\ell^2}$$

$$N = \int d\xi\, e^{-\frac{1}{2A}\delta t \xi^2}$$

$$\langle O \rangle = \int \mathcal{D}(x)\mathcal{D}(y)\, e^{\underbrace{\int dt\left[-iy(\partial_t x - f(x)) - \frac{A g(x)^2}{2} y^2\right]}_{=iS}}$$

$$\mathcal{D}(x) = \prod_i dx_i \qquad \mathcal{D}(y) = \prod_i \frac{dy_i}{2\pi}$$

$$\int dt[\dots] = \sum_\ell \left[-i y_\ell (x_\ell - x_{\ell-1} - \delta t f(x_{\ell-1})) - \frac{A \delta t\, y_\ell^2 g(x_{\ell-1})^2}{2}\right]$$

understood in Itō sense

(Martin - Siggia - Rose -]anssen - de Dominics (MSRJD) functional integral)

to get conditional probability $p(xt, x't')$, $\begin{pmatrix} t = n\delta t \\ t' = 0 \end{pmatrix}$

set $O(x) = \delta(x_n - x)\,\delta(x_0 - x')$

$$p(xt, x't') = \int_{x't'}^{xt} \mathcal{D}(x)\mathcal{D}(y)\, e^{\int dt[\dots]}$$

often convenient representation: $y_\ell = -i p_\ell$

$$\mathcal{D}(y) = \prod_i \int_{-\infty}^{+\infty} dy_i \to \prod_\ell \int_{-\infty i}^{+i\infty} dp_\ell = \mathcal{D}(p)$$

$$i S[x,p] = -\int dt\,[p\dot{x} - H(x,p)]$$

$$H[x,p] = p f(x) + p^2 g(x)^2 \frac{A}{2}$$

looks like statistical path integral in (imaginary time) quantum mechanics! But p has different meaning!

optimal path approximation

For some cases (see below), good approximation of path integral with stationary phase approx. possible ($\hat{=}$ corresponds to the "classical dynamics" belonging to the pseudo Hamiltonian $H(x,p)$)

Stationary-phase approximation

evaluate complex integral $I(N) = \int_C dz\, e^{N f(z)}$ with analytic function $f(z)$ for large $N(\in \mathbb{R})$?

$\rightsquigarrow$ assume that there is a "stationary point": $\left.\frac{df}{dz}\right|_{z_0} = 0$, so that close to $z = z_0$, we have

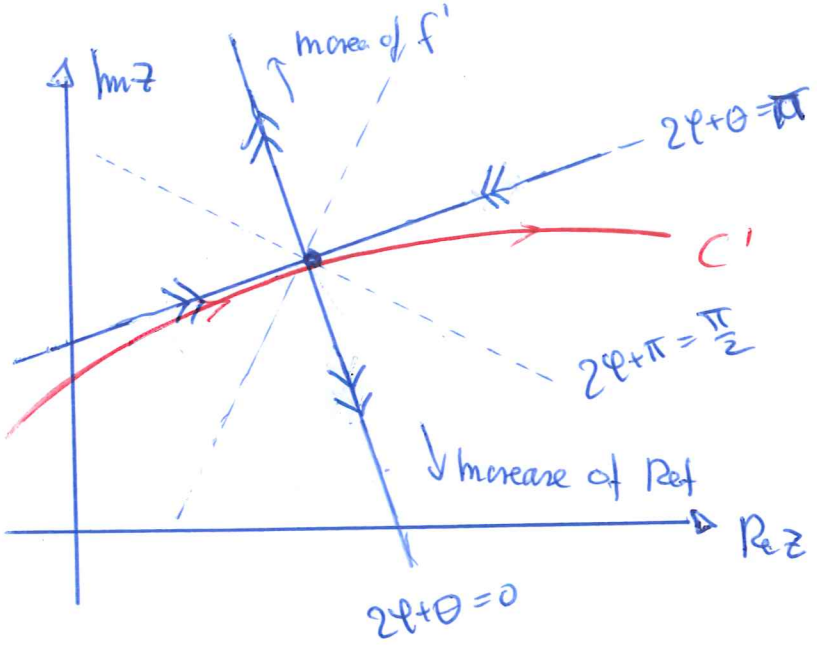
$$f(z) = f(z_0) + \frac{a}{2}(z - z_0)^2 + \dots$$

for $z \to z_0$: set $z - z_0 = r e^{i\varphi}$ $a = |a| e^{i\theta}$

$$f(z) = f(z_0) + \frac{r^2 |a|}{2} e^{2i\varphi + i\theta} + \dots$$

along direction

- $2\varphi + \theta = 0$: Ref has min , $\mathrm{Im} f \approx$ const.
- $2\varphi + \theta = \frac{\pi}{2}$: $\mathrm{Re} f \simeq$ const , $\mathrm{Im} f \simeq$ min
- $2\varphi + \theta = \pi$: $\mathrm{Re} f \simeq$ max $\mathrm{Im} f \simeq$ const
- $2\varphi + \theta = \frac{3\pi}{2}$: $\mathrm{Re} f \simeq$ const $\mathrm{Im} f \simeq$ max



⇝ deform $C \to C'$ such that it passes along direction $2\varphi + \theta = \pi$

⇝ contribution to $I(N)$ from region $z \approx z_0$

$$\int_{C'} dz\, e^{N f(z)} \simeq \pm \int dr\, e^{i(\pi-\theta)/2}\, e^{-\frac{N|a|}{2} r^2}\, e^{N f(z_0)}$$

direction of C' given by direction of C

can extend to $(-\infty, +\infty)$ for large N, with exp. small error.

assume that C' can be chosen such that z_0 is not only local, but also global maximum of $\mathrm{Re}\, f$ (in particular, no maximum at boundaries, like for $\int_{-1}^{1} dx\, e^{x^2 N}$)

$$I(N) \underset{N\to\infty}{\sim} \pm e^{i\left(\frac{\pi-\Theta}{2}\right)} \frac{1}{\sqrt{|f''(z_0)N|}} e^{N f(z_0)}$$

often only the term $e^{N f(z_0)}$ matters, as it includes the most important exponential dependence of the result on the ~~prefa~~ parameters.

Application to path integral

Brute force stat. phase approx. to ~~in~~ each dx_i, dp_i in MSRJD path integral

$$\int_{x_i t_i}^{x_f t_f} \mathcal{D}(x)\mathcal{D}(p)\, e^{i S[x,p]} \simeq e^{i S_0}$$

↙ sub-dominant prefactor omitted

$S_0 = S[\tilde{x}, \tilde{p}]$, where $\tilde{x}(t)$, $\tilde{p}(t)$ determined by stationary action $\frac{\delta S}{\delta x} = 0$, $\frac{\delta S}{\delta p} = 0$

with $iS = -\int dt\,[p\dot{x} - H(x,p)]$

$$\frac{\delta S}{\delta x} = 0 \Rightarrow 0 \overset{!}{=} \int dt\,[p \cdot \tfrac{d}{dt}\delta x - \partial_x H(x,p)\,\delta x]$$

$$= \int dt\,[(-\dot{p} - \partial_x H(x,p))\,\delta x] = 0$$

$$\underbrace{\quad}_{\overset{!}{=}0}$$

$$\frac{\delta S}{\delta p} = 0 \Rightarrow 0 \overset{!}{=} \int dt\,\delta p\,(\underbrace{\dot{x} - \partial_p H}_{\overset{!}{=}0})$$

$$\Rightarrow \boxed{\begin{aligned} \dot{\tilde{x}} &= \partial_p H(\tilde{x},\tilde{p}) = f(x) + A\,g(x)^2 p \\ \dot{\tilde{p}} &= -\partial_x H(\tilde{x},\tilde{p}) = -p f'(x) - A\,p^2 g(x) g'(x) \end{aligned}}$$

Hamilton equations of motion!

Validity → where is large parameter "N" ?

Example: particle in potential $V(x)$:

$$m\ddot{x} = -\gamma \dot{x} - V'(x) + \xi(t)$$

$$\langle \xi(t)\, \xi(t') \rangle = 2\gamma T\, \delta(t-t')$$

consider "overdamped limit" $|m\ddot{x}| \ll |\gamma \dot{x}|$

$$\gamma \dot{x} = -V'(x) + \xi(t)$$

$\hat{=}$ generalized Langevin eq. with

$$f(x) = -V'(x)/\gamma \quad g(x) = 1/\gamma \quad A = 2T\gamma$$

$\rightsquigarrow$ "Hamiltonian": $H(x,p) = -\frac{V'(x)}{\gamma} p + \frac{p^2 T}{\gamma}$

$$\dot{x} = -\frac{V'(x)}{\gamma} + \frac{Tp}{\gamma} \qquad \dot{p} = \frac{V''(x)}{\gamma} p$$

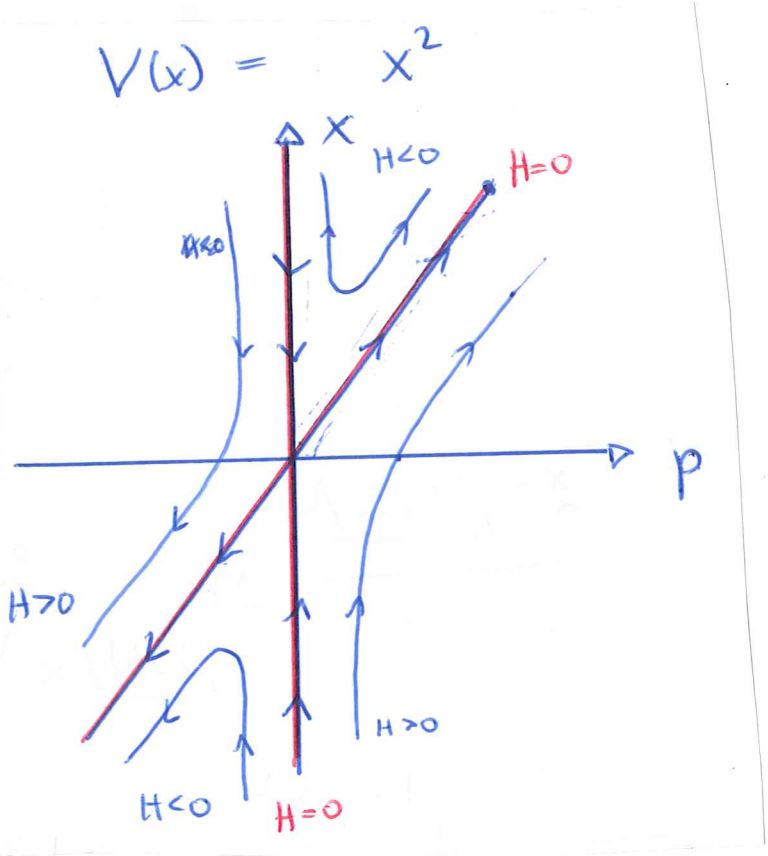
rescaling time $\tau = \gamma t$

$\gamma t \to t \qquad H/\gamma \to H$ (time measured in units of $1/\gamma$)

$$\boxed{\dot{x} = -V'(x) + Tp \qquad \dot{p} = V''(x) \qquad H = -V'(x)p + p^2 T}$$

because $H(x,p)$ does not depend on time, equations of motion conserve $H(x,p) \equiv E$, i.e. $(x(t), p(t))$ follows const-H surfaces.

phase portrait for $V(x) = x^2$



$E = 0 \rightsquigarrow$

$\begin{cases} p = 0 & \text{or} \\ p = \frac{V'(x)}{T} = x/T \end{cases}$

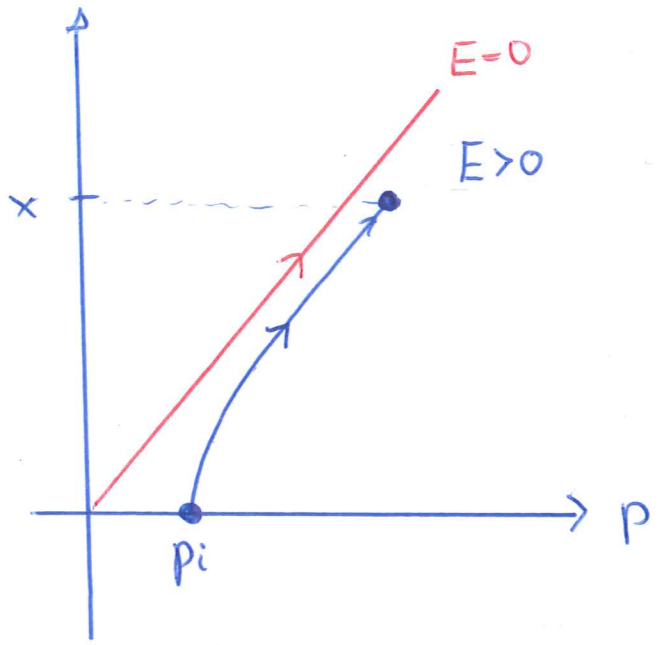
probability to find particle at (x,t), while is initially at $x=0$ for $t = t_i \to -\infty$:

$$W(x,t) \sim e^{i S_0} \qquad i S_0 = -\int dt\,(p\dot{x} - H)$$

along trajectory $x=0, t=t_i \longrightarrow x, t$

initial "momentum" p_i ?

$$\left\{ \begin{array}{l} \text{time along trajectory} \\ p_i \; x=0, t_i \longrightarrow x \end{array} \right\} = \int_0^x dx' \frac{1}{\dot{x}(x; p_i)} \stackrel{!}{=} t - t_i$$



$\dot{x} = 0$ ou $E = 0$ trajectory

$\Rightarrow$ for $t_i \to -\infty$ $p_i \to$ ~~xxxx~~ 0

$$i S_0 = -\int dt \, (p\dot{x} - E) = -\int_0^x p \, dx$$

$\uparrow V'(x)$ / T

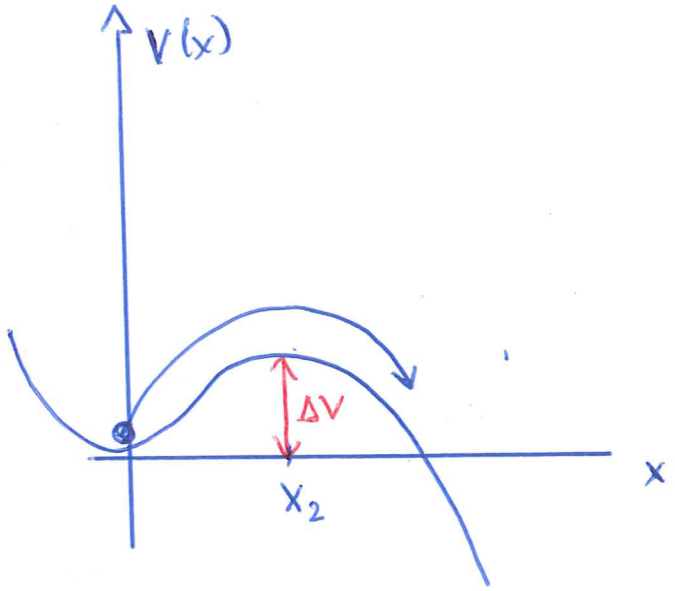
$$= -\frac{V(x)}{T}$$

$$\Rightarrow W(x,t) \sim e^{-V(x)/T}$$

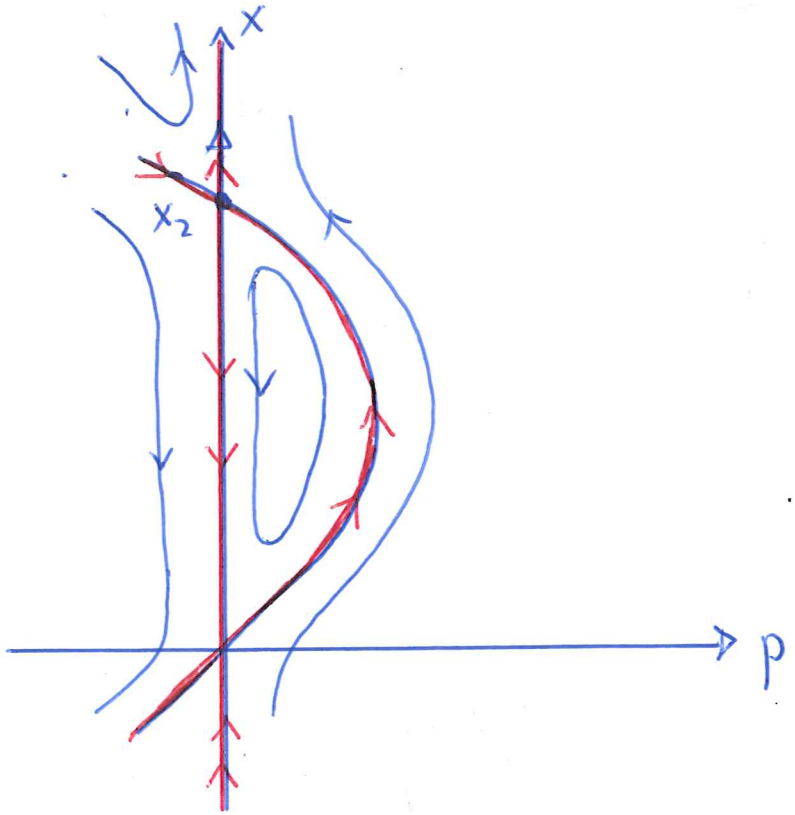
Boltzmann distribution → total overkill

... slightly more interesting ...

escape of particle from metastable minimum:

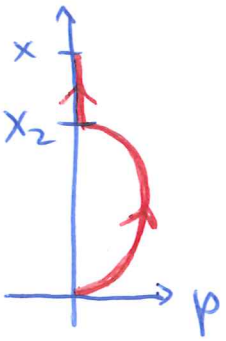


unstable point at $x = x_2$ ⇒ opposite structure in phase-space compared to stable one at $x = 0$.



probability to find particle at some point $x > x_2$

$\sim e^{iS_0}$ iS_0: action along $E = 0$ trajectory



analogous as above ...

... only path ② does not contribute to S_0 ($p=0$ in $\int \dot{x} p dt$)

$$\Rightarrow \quad i S_0 = - \int_{x=0}^{x_2} dx' V'(x') / T = - \frac{V(x_2) - V(0)}{T}$$

$$W \sim e^{- \frac{V(x_2) - V(0)}{T}} = e^{- \Delta V / T} \quad //$$